UNIVERSAL PROPERTY OF TRIANGULATED DERIVATORS VIA KELLER'S TOWERS

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Abstract. In [9] B. Keller solved the universal problem of the extension of an exact category to its (bounded) derived category by introducing the notions of tower of exact and triangulated categories and proving the universal property in this setting. In this note we show that his result extends to the corresponding universal problem for Grothendieck's derivators.

Contents

1. Introduction

Triangulated categories were invented in the early sixties by Grothendieck–Verdier [16] and, simultaneously and independently, by Puppe. Grothendieck–Verdier sought to axiomatize the properties of derived categories of sheaves, while Puppe was motivated by examples from topology, and notably the stable homotopy category of finite CW-complexes.

In spite of the success of derived and triangulated categories in recent years, during the thirty years that followed, triangulated categories were largely considered to be too poor in structure to allow the development of a significant general theory, analogous to that of abelian categories established by Freyd, Mitchell, Gabriel, ... As an example, let us consider a triangulated category $\mathcal{T}$ and let I be a finite directed diagram (*i.e.*, a

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small finite category with no nontrivial loops). We can form the category $\mathsf{Hom}(I^\circ, \mathcal{T})$ of contravariant functors from the diagram I into $\mathcal{T}$. In general, there is no canonical triangulated structure on this category, even in the simple case where I is the category containing only two objects, their identities, and an arrow connecting them, the main problem being that the mapping cone is a non-functorial construction. Another important consequence of this fact is the following: Let $\mathcal{E}$ be an exact category and $\mathcal{D}^b(\mathcal{E})$ its bounded derived category. Then, the universal property does not hold for the triangulated category $\mathcal{D}^b(\mathcal{E})$, *i.e.*, the natural functor $\mathcal{H}om_{ex}(\mathcal{D}^b(\mathcal{E}), \mathcal{T}) \to \mathcal{H}om_{ex}(\mathcal{E}, \mathcal{T})$, relating categories of exact functors, is not quasi-invertible for all triangulated categories $\mathcal{T}$.

It follows that, in order to have a satisfactory theory, the notion of triangulated category must be modified or enhanced. A long list of attempts appeared in the literature in recent years in order to define new kinds of structures assuring the universal property to the construction $\mathcal{D}^b$, *e.g.* , DG-categories and $\mathcal{A}_\infty$-categories. Derivators were introduced by Alexander Grothendieck [6] in the nineties in a long letter addressed to Quillen. Their advantage, compared to the other constructions, is that they have the least amount of added structure. I will not try to give a complete definition of what a derivator is in this introduction. It suffices to say that it is a 2-functor $\mathbb{D}$ from the 2-category of diagrams into the 2-category of categories $\mathcal{CAT}$ satisfying a list of axioms inspired by Quillen's definition of a model category [15] together with its homotopy category.

Similar constructions were independently introduced by A. Heller and B. Keller in the nineties. Keller's construction [9], called 'tower', only considers categories 'indexed' over hyper-cubical diagrams, but nevertheless he was able to prove the universal property for the derived category. Some years later, G. Maltsiniotis [13], principally inspired by works of Grothendieck and Franke, gave the axioms for derivators that we follow in this paper. He introduced a notion of triangulated derivator, *i.e.*, a derivator taking values into triangulated categories and satisfying two more axioms [13]. At the same time [10], Keller associates a derivator $\mathbb{D}_\mathcal{E}$ with an exact category $\mathcal{E}$ by means of the following definition: $\mathbb{D}_\mathcal{E}(I) := \mathcal{D}^b(\underline{\mathsf{Hom}}(I^\circ, \mathcal{E}))$, and proves that this derivator is triangulated.

The principal aim of this paper is to show that, similarly to the case of 'towers', the triangulated derivator associated to an exact category $\mathcal{E}$ has a universal property. This is Theorem 2.13 in the text. As a corollary, in the particular case of the diagram $I = e$ (the one object category), we obtain the correct universal property for the (bounded) derived category $\mathbb{D}_\mathcal{E}(e) = \mathcal{D}^b(\mathcal{E})$ (Theorem 2.14). This result is similar in spirit to a string of other results about universal properties of derivator-like structures obtained by other authors. A. Heller proved his main results on 'homotopy theories', a notion very near to Grothendieck's notion of 'dérivateur', in [7] already in the '80s. J. Franke introduced in 1996 his notion of 'systems of triangulated diagram categories' in [4]. Both these authors proved universal properties for the structures they have introduced, the main difference being that in the former the base is given by the homotopy category of simplicial sets of arbitrary size rather than of finite spectra as in the latter. We summarize and report these historical results in Theorem 2.16 in the text. Finally, in 2008 D.-C. Cisinski [2] proved a universal property for the derivator $\mathbb{H}\mathrm{ot}_I$ associated to the homotopy theory of presheaves over a diagram I with values in small categories.

Let us briefly describe the contents of the sections of the present article. In section 2 we fix the notations and recall the definitions of derivator, pointed derivator, triangulated derivator, morphism of derivators. Then, after recalling the notion of exact category, we introduce the associated triangulated derivator defined by Keller in [10], which is the central object of interest in this paper. Section 3 is dedicated to the description of the notion of Keller's tower and its related morphisms. We prove our main result about the universal

property of this derivator in section 5. Let us remark that one could write down a proof completely expressed in the language of derivators, following the lines of Keller's proof. However, in this paper we prefer to show how the theorem in the setting of derivators naturally follows from the case of towers.

Section 4 is devoted to a very useful property of morphisms of derivators that we call, following Keller's [9], 'redundancy of the connecting morphism'. This means that a morphism of derivators which preserves bicartesian squares (in the sense of derivators) authomatically preserves connecting morphisms, hence preserves distinguished triangles. This fact entails that an additive morphisms of exact or triangulated derivators F is authomatically triangulated provided that it locally preserves distinguished triangles, even when it does not preserves connecting morphisms functorially. The proof involves the formalism said 'recollement of triangulated categories' or '6-gluing functors' for low dimensional diagrams.

The proof of the universal property for (bounded) derived categories is left to section 6. The techniques used in the proof are new with respect to Keller's approach in [9]. Indeed, Keller could prove his result using induction over the integer dimensions of his diagrams since he was dealing with hypercubes. This is not possible if we consider all finite directed diagrams (*cf.* definition in Section 2) and we have to change our strategy. A key ingredient in our proof is a careful analysis of the property of epivalence (*i.e.*, essential surjectivity and fullness) of the diagram functor dia_I, for any I, which sends an object X in the category $\mathbb{T}(I)$ to its 'diagram' in the base $\mathbb{T}(e)$. This is done in subsections 6.1 and 6.2 under the hypothesis that is commonly known in the literature as 'Toda condition'.

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2. Grothendieck's derivators

In this section we briefly recall the definition of derivator in the sense of Grothendieck, following the exposition in [13]. The reader is invited to look at the original manuscript [6] for a complete exposition. We begin by reminding some useful categorical notions and fixing the notations.

We denote by $\mathscr{C}at$ the 2-category of small categories. Among its objects there are $\varnothing$, the empty 1-category, and e, the 1-category with one object $*$ and only the identity morphism. If $\mathcal{A}$ is a small 1-category, we write $\mathcal{A}^\circ$ to indicate the opposite 1-category. Given two 1-categories $\mathcal{A}$ and $\mathcal{B}$, then $\underline{\mathsf{Hom}}(\mathcal{A},\mathcal{B})$ denotes the 1-category of functors (1-morphisms in $\mathscr{C}at$) from $\mathcal{A}$ into $\mathcal{B}$ with natural transformations as morphisms (2-morphisms in $\mathscr{C}at$). For any functor $u : \mathcal{A} \to \mathcal{B}$ and a fixed object b in the category $\mathcal{B}$, the category $\mathcal{A}/b$ contains the pairs (a, f), where a is an object of $\mathcal{A}$ and f a morphism from $u(a)$ into b in the category $\mathcal{B}$. A morphism $\varphi : (a, f) \to (a', f')$ in the category $\mathcal{A}/b$ is determined by an arrow $g : a \to a'$ such that $f' \circ u(g) = f$. Composition of morphisms in $\mathcal{A}/b$ is clearly induced by composition in $\mathcal{A}$. Dually, we have the category $b\backslash\mathcal{A}$ defined by $b\backslash\mathcal{A} = (\mathcal{A}^\circ/b)^\circ$. There are canonical forgetful functors from $\mathcal{A}/b$ and $b\backslash\mathcal{A}$ into $\mathcal{A}$, defined by $(a, f) \mapsto a$. By considering the identity functor of $\mathcal{B}$, we can canonically associate the categories $\mathcal{B}/b$ and $b\backslash\mathcal{B}$ with $\mathcal{B}$, for any arbitrarily fixed object $b \in \mathcal{B}$. Let us call $u/b : \mathcal{A}/b \to \mathcal{B}/b$ and $b\backslash u : b\backslash\mathcal{A} \to b\backslash\mathcal{B}$ the induced functors which associate the pair (a, f) with the pair $(u(a), f)$. It is easy to check that the following commutative squares are cartesian

$$
\begin{tikzcd}
\mathcal{A}/b \arrow[r] \arrow[d, "u/b"'] & \mathcal{A} \arrow[d, "u"] \\
\mathcal{B}/b \arrow[r] & \mathcal{B}
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
b\backslash\mathcal{A} \arrow[r] \arrow[d, "b\backslash u"'] & \mathcal{A} \arrow[d, "u"] \\
b\backslash\mathcal{B} \arrow[r] & \mathcal{B}
\end{tikzcd}
\quad .
$$

Let us name $\mathscr{D}ia_{\mathsf{f}}$ the full 2-subcategory of $\mathscr{C}at$ whose objects are finite directed categories, *i.e.*, categories whose nerve has only a finite number of non-degenerate simplices. Equivalently, we can say that the objects in $\mathscr{D}ia_{\mathsf{f}}$, that we call *(finite) diagrams*, are the finite categories whose underlying quiver (vertices: objects, arrows: non identical morphisms) does not have oriented cycles (*e.g.* , finite posets). These objects, together with the functors of categories as 1-morphisms and the natural transformations as 2-morphisms, endow $\mathscr{D}ia_{\mathsf{f}}$ with the structure of a 2-category.

There are more general categories of diagrams that can be used in the definition of a derivator (see, *e.g.* , [13] for a discussion and references therein). We simply write $\mathscr{D}ia$ whenever it is possible to work in those wider situations.

Definition 2.1. A *prederivator of type $\mathscr{D}ia$* is a strict 2-functor

$$\mathbb{D} : \mathscr{D}ia^{\circ} \longrightarrow \mathscr{C}at.$$

More explicitly, this means that there are a category $\mathbb{D}(I)$ associated with any diagram I, a functor $u^* := \mathbb{D}(u)$, $u^* : \mathbb{D}(J) \to \mathbb{D}(I)$, associated with any functor $u : I \to J$, and a natural transformation $\alpha^* := \mathbb{D}(\alpha)$,

$$\begin{tikzcd} \mathbb{D}(J) \arrow[r, bend left, "u^*"] \arrow[r, bend right, "v^*"'] \arrow[r, phantom, "\alpha^*\Uparrow"] & \mathbb{D}(I) \end{tikzcd} ,$$

associated with any natural transformation

$$\begin{tikzcd} I \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & J \end{tikzcd} .$$

These data have to verify the following coherence axioms:

- $\mathbf{1}_I^* = \mathbf{1}_{\mathbb{D}(I)}$, for any object I in $\mathscr{D}ia$;
- $\mathbf{1}_u^* = \mathbf{1}_{u^*}$, for any arrow $I \overset{u}{\to} J$ in $\mathscr{D}ia$;
- $(vu)^* = u^*v^*$, for any diagram $I \overset{u}{\to} J \overset{v}{\to} K$ in $\mathscr{D}ia$;
- $(\beta\alpha)^* = \alpha^*\beta^*$, for any diagram $\begin{tikzcd} I \arrow[r, bend left=60, "u"] \arrow[r, "v" description] \arrow[r, bend right=60, "w"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & J \end{tikzcd}$ in $\mathscr{D}ia$;
- $(\beta \star \alpha)^* = \alpha^* \star \beta^*$, for any diagram $\begin{tikzcd} I \arrow[r, bend left, "u"] \arrow[r, bend right, "u'"'] \arrow[r, phantom, "\Downarrow\alpha"] & J \arrow[r, bend left, "v"] \arrow[r, bend right, "v'"'] \arrow[r, phantom, "\Downarrow\beta"] & K \end{tikzcd}$ in $\mathscr{D}ia$.

If $u : I \to J$ is a (1-)morphism in $\mathscr{D}ia$, we respectively denote u_* and $u_!$ the right and left adjoint functor to u^*, when they exist.

2.1. **Triangulated derivators.** We have to introduce some more notations and terminology in order to define the notion of derivator.

For any object x of a small category I lying in $\mathscr{D}ia$, we write $i_{x,I} : e \to I$ to indicate the functor that is uniquely determined by x and I. Sometimes, we will simply write i_x, or even x, when the context is clear. Given an arbitrary prederivator $\mathbb{D}$ and an object F of $\mathbb{D}(I)$, the object $F_x := i_x^*(F)$ is called the *fiber of F at the point x*. If $\varphi : F \to F'$ is a morphism in $\mathbb{D}(I)$, the morphism $\varphi_x := i_x^*(\varphi) : F_x \to F'_x$ is the *morphism of fibers induced by φ*.

For any 2-square in $\mathscr{D}ia$,

$$\mathcal{D} \quad = \quad \begin{tikzcd} I' \arrow[r, "v"] \arrow[d, "u'"'] \arrow[dr, phantom, "\swarrow\alpha"] & I \arrow[d, "u"] \\ J' \arrow[r, "w"'] & J \end{tikzcd} \qquad\qquad \alpha : uv \longrightarrow wu' \ ,$$

we denote by $\varepsilon : u^*u_* \to \mathbf{1}_{\mathbb{D}(I)}$, $\eta : \mathbf{1}_{\mathbb{D}(J)} \to u_*u^*$, $\varepsilon' : u'^*u'_* \to \mathbf{1}_{\mathbb{D}(I')}$, $\eta' : \mathbf{1}_{\mathbb{D}(J')} \to u'_*u'^*$ the adjunction morphisms. Let us define the *base change morphism* $c_{\mathcal{D}} : w^*u_* \to u'_*v^*$ to be the composition $(u'_*v^* \star \varepsilon)(u'_* \star \alpha^* \star u_*)(\eta' \star w^*u_*)$ of the following morphisms

$$w^*u_* \xrightarrow{\ \eta' \star w^* u_*\ } u'_*u'^*w^*u_* = u'_*(wu')^*u_* \xrightarrow{\ u'_*\star\alpha^*\star u_*\ } u'_*(uv)^*u_* = u'_*v^*u^*u_* \xrightarrow{\ u'_*v^*\star\varepsilon\ } u'_*v^*.$$

Clearly, we have a dual morphism $c'_{\mathcal{D}} : v_!u'^* \to u^*w_!$.

Given a small category I in $\mathscr{D}ia$, we indicate as $p_I : I \to e$ the unique functor from I into the one object category e. In order to remain compatible with the standard notations of model category theory, we can set, for any object F in $\mathbb{D}(I)$,

$$\underleftarrow{\mathsf{holim}}_I F \coloneqq (p_I)_*(F) \qquad \text{and} \qquad \underrightarrow{\mathsf{holim}}_I F \coloneqq (p_I)_!(F)$$

in the category $\mathbb{D}(e)$ and talk about *homotopical projective and inductive limit* of F.

We remark that sometimes the notations $\Gamma_*(I,F) \coloneqq (p_I)_*(F)$ and $\Gamma_!(I,F) \coloneqq (p_I)_!(F)$ are also used. In this way, these objects can be thought of as the *global sections of F over I*. The topologically inclined reader might speak of the *(co-)homology of I with coefficients in F*. Other common notations that we also use in this paper are $\mathsf{holim}_I F$ and $\mathsf{hocolim}_I F$, respectively for $\underleftarrow{\mathsf{holim}}_I F$ and $\underrightarrow{\mathsf{holim}}_I F$.

Note that the notions of homotopy limit and fiber of an object F are not completely unrelated. Indeed, we can construct a comparison morphism in the following way. Let $u : I \to J$ be a morphism in $\mathscr{D}ia$, y an object of J and F an object in $\mathbb{D}(I)$. There are the forgetful functors $j : I/y \to I$ and $j : y\backslash I \to J$. We denote by $F|_{I/y}$ (resp. $F|_{y\backslash I}$) the image $j^*(F)$ in $\mathbb{D}(I/y)$ (resp. in $\mathbb{D}(y\backslash I)$). Let us consider the 2-squares

$$\mathcal{D}_{u/y} = \begin{tikzcd} I/y \arrow[r, "j"] \arrow[d, "p_{I/y}"'] \arrow[dr, phantom, "\swarrow\alpha"] & I \arrow[d, "u"] \\ e \arrow[r, "y"'] & J \end{tikzcd} \qquad \text{and} \qquad \mathcal{D}_{y\backslash u} = \begin{tikzcd} y\backslash I \arrow[r, "j"] \arrow[d, "p_{y\backslash I}"'] \arrow[dr, phantom, "\alpha'\nearrow"] & I \arrow[d, "u"] \\ e \arrow[r, "y"'] & J \end{tikzcd} \ ,$$

where the 2-morphisms α and α' are defined, for any object x of I, by the formulae $\alpha_{(x,\, f:u(x)\to y)} \coloneqq f$ and $\alpha'_{(x,\, g:y\to u(x))} \coloneqq g$. For any F in $\mathbb{D}(I)$, there are the associated base change canonical morphisms

$$c_{\mathcal{D}_{u/y}} : \ (u_*F)_y \longrightarrow (p_{I/y})_*j^*(F) = \underleftarrow{\mathsf{holim}}_{I/y}(F|_{I/y}) \qquad \text{and}$$

$$c'_{\mathcal{D}_{y\backslash u}} : \ \underrightarrow{\mathsf{holim}}_{y\backslash I}(F|_{y\backslash I}) = (p_{y\backslash I})_!j^*(F) \longrightarrow (u_!F)_y \qquad .$$

Suppose that we have fixed a prederivator $\mathbb{D}$. Let us consider two arbitrary diagrams I, J in $\mathscr{D}ia$. For any object x of I, there is the canonical functor $x_{I,J} : J \to I \times J$, which associates any object y in the category J with the pair (x,y). By the 2-functoriality of $\mathbb{D}$, there exists

the functor $x_{I,J}^{*} : \mathbb{D}(I \times J) \to \mathbb{D}(J)$ and a morphism of functors $\alpha_{I,J}{}^{*} : x'_{I,J}{}^{*} \to x_{I,J}{}^{*}$, for every morphism $\alpha : x \to x'$ in I. In other words, we have a functor of 1-categories

$$\mathbb{D}(I \times J) \times I^{\circ} \longrightarrow \mathbb{D}(J)\,,$$

which associates the object $x_{I,J}{}^{*}(F)$ of $\mathbb{D}(J)$ with a pair (F, x). By adjunction, we get the functor

$$\mathsf{dia}_{I,J} : \mathbb{D}(I \times J) \longrightarrow \underline{\mathsf{Hom}}(I^{\circ}, \mathbb{D}(J))\,,$$

where $\underline{\mathsf{Hom}}(I^{\circ}, \mathbb{D}(J))$ denotes the 1-category of contravariant functors from I into $\mathbb{D}(J)$. When $J = e$, the functor $\mathsf{dia}_{I,e}$ induces a functor

$$\mathsf{dia}_{I} : \mathbb{D}(I) \longrightarrow \underline{\mathsf{Hom}}(I^{\circ}, \mathbb{D}(e))\,,$$

whose image $\mathsf{dia}_I(F)$, for any F in $\mathbb{D}(I)$, is called *underlying diagram of* F or simply *diagram of* F. It is the presheaf

$$\mathsf{dia}(F) \coloneqq \mathsf{dia}_I(F) : I^{\circ} \longrightarrow \mathbb{D}(e)$$

defined by the equality $\mathsf{dia}_I(F)(x) = F_x$, for any object x in I.

Definition 2.2. A *(Grothendieck's) derivator of type* $\mathscr{D}ia$ is a prederivator $\mathbb{D}$ of type $\mathscr{D}ia$ which satisfies the following axioms.

Der 1 a) If I and J are in $\mathscr{D}ia$, the functor

$$\mathbb{D}(I \textstyle\coprod J) \xrightarrow{(i^*, j^*)} \mathbb{D}(I) \times \mathbb{D}(J)\,,$$

induced by the canonical functors $i : I \to I \coprod J$ and $j : J \to I \coprod J$, is an equivalence of 1-categories;

b) the category $\mathbb{D}(\varnothing)$ is equivalent to the point category e.

Der 2 For any category I in $\mathscr{D}ia$, the family of functors $i_x^{*} : \mathbb{D}(I) \to \mathbb{D}(e)$, indexed by the objects x lying in I, is conservative. Explicitly, this means that any morphism $\varphi : F \to F'$ in $\mathbb{D}(I)$ is an isomorphism iff the morphism of fibers $\varphi_x = i_x^{*}(f) : F_x \to F'_x$ is an isomorphism in $\mathbb{D}(e)$.

Der 3 For any morphism $u : I \to J$ in $\mathscr{D}ia$, the induced functor $u^{*} : \mathbb{D}(J) \to \mathbb{D}(I)$ admits a right adjoint $u_{*} : \mathbb{D}(I) \to \mathbb{D}(J)$ and a left adjoint $u_{!} : \mathbb{D}(I) \to \mathbb{D}(J)$.

Der 4 For any morphism $u : I \to J$ in $\mathscr{D}ia$, any object y in J and any F in $\mathbb{D}(I)$, the associated base change canonical morphisms

$$c_{\mathcal{D}_{u/y}} :\ (u_* F)_y \longrightarrow (p_{I/y})_* j^*(F) = \underleftarrow{\mathsf{holim}}_{I/y}(F|_{I/y}) \qquad \text{and}$$

$$c'_{\mathcal{D}_{y\backslash u}} :\ \underrightarrow{\mathsf{holim}}_{y\backslash I}(F|_{y\backslash I}) = (p_{y\backslash I})_! j^*(F) \longrightarrow (u_! F)_y$$

are invertible.

Der 5 (*Epivalence axiom*). *For any J in $\mathscr{D}ia$, the functor*

$$\mathsf{dia}_{\Delta_1, J} : \mathbb{D}(\Delta_1 \times J) \longrightarrow \underline{\mathsf{Hom}}(\Delta_1^{\circ}, \mathbb{D}(J))\,,$$

where Δ_1 denotes the category $\{0 \leftarrow 1\}$ of $\mathscr{D}ia$, is full and essentially surjective.

We recall that some authors neglect Axiom **Der 5** in the definition of a derivator and reserve the name 'strong derivator' for our notion of derivator. Nevertheless, we prefer to include this axiom since we are above all interested in *triangulated* derivators and the epivalence axiom is central for the construction of the triangulated structure of the categories $\mathbb{D}(I)$, as in the proof of Theorem 2.6.

Let us remark that the axioms above assure us that the category $\mathbb{D}(I)$ has products and coproducts indexed over the sets of morphisms and objects of any diagram I contained in $\mathscr{D}ia$.

A 1-morphism $j : U \to I$ of $\mathscr{D}ia$ is an *open immersion* if it is injective on objects, fully faithful, and if any morphism $f : y \to j(x)$ in I is in the image of j, *i.e.*, it is of the form $j(g) : j(x') \to j(x)$, for some morphism $g : x' \to x$ in U. Dually, a 1-morphism $i : Z \to I$ of $\mathscr{D}ia$ is a *closed immersion* if $i^\circ : Z^\circ \to I^\circ$ is an open immersion. Open immersions and closed immersions are stable under composition and pullback.

Definition 2.3. A derivator $\mathbb{D}$ is *pointed* if the following axiom holds.

Der 6 For any closed immersion $i : Z \to I$ in $\mathscr{D}ia$, the induced functor i_* admits a right adjoint $i^!$. Dually, for any open immersion $j : U \to I$ in $\mathscr{D}ia$, the induced functor $j_!$ admits a left adjoint $j^?$.

Coming back to notations and terminology, we write $\square$ to indicate the category $\Delta_1 \times \Delta_1$, $\ulcorner$, $\lrcorner$ to indicate the two subcategories

$$
\begin{tikzcd}
(0,0) & (0,1) \arrow[l] \\
(1,0) \arrow[u] &
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
 & (0,1) \\
(1,0) & (1,1) \arrow[l] \arrow[u]
\end{tikzcd}
$$

of $\square$, and $i_\ulcorner : \ulcorner \to \square$ $i_\lrcorner : \lrcorner \to \square$ to indicate the inclusion functors. An object F of $\mathbb{D}(\square)$ is *homotopically cartesian* (resp., *homotopically cocartesian*), or, more simply, *cartesian* (resp., *cocartesian*), if the adjunction morphism

$$\eta^F : F \longrightarrow (i_\lrcorner)_*(i_\lrcorner)^* F \qquad \text{(resp., } \varepsilon^F : (i_\ulcorner)_!(i_\ulcorner)^* F \longrightarrow F)$$

is an isomorphism.

Definition 2.4. A pointed derivator is *triangulated* if the following axiom holds.

Der 7 Any object F in $\mathbb{D}(\square)$ is homotopically cartesian if and only if it is homotopically cocartesian (*i.e.*, homotopically bicartesian).

Example 2.5. Let $\mathcal{M}$ be a category such that there exists a stable Quillen closed model structure with $\mathcal{W}$ as weak equivalences. Then, the prederivator $\mathbb{D}_{(\mathcal{M},\mathcal{W})}$ defined, on any small category I, by

$$\mathbb{D}_{(\mathcal{M},\mathcal{W})}(I) := \underline{\mathsf{Hom}}(I^\circ, \mathcal{M})[\mathcal{W}^{-1}]$$

and, on any functor $u : I \to J$, by

$$\mathbb{D}_{(\mathcal{M},\mathcal{W})}(J) \xrightarrow{u^* := \overline{(u^\circ)^*}} \mathbb{D}_{(\mathcal{M},\mathcal{W})}(I) \ ,$$

is a triangulated derivator. Here the overline means the induced functor between the localized categories.

For example, the stable model category of spectra gives rise to a triangulated derivator $\mathbb{S}p$. The following theorem was announced in [13]. The interested reader can find a proof in [5, Theorem 4.15].

Theorem 2.6 (Maltsiniotis)**.** *If $\mathbb{D}$ is a triangulated derivator, for every I in $\mathscr{D}ia$, there is a canonical structure of triangulated category on $\mathbb{D}(I)$, such that for every 1-morphism $u : I \to J$ in $\mathscr{D}ia$, the functor $u^* : \mathbb{D}(J) \to \mathbb{D}(I)$ is canonically endowed with the structure of triangulated functor.*

2.2. Morphisms of derivators.

Definition 2.7. Let $\mathbb{D}$ and $\mathbb{E}$ be two prederivators. A *morphism* (*i.e.*, a 1-morphism) $F : \mathbb{D} \to \mathbb{E}$ is given by

- a functor $F_I : \mathbb{D}(I) \to \mathbb{E}(I)$, for each I in $\mathscr{D}ia$;
- an isomorphism of functors
$$\varphi_u : F_I u^* \tilde{\to} u^* F_J \ ,$$
for each 1-morphism $u : I \to J$ in $\mathscr{D}ia$;

such that

- $\varphi_{\mathbf{1}_I} = \mathbf{1}_{F_I}$, for each I in $\mathscr{D}ia$;
- $\varphi_{uv} = (v^* \varphi_u)(\varphi_v u^*)$, for each pair of 1-morphisms $K \overset{v}{\to} I \overset{u}{\to} J$ in $\mathscr{D}ia$;
- $\varphi_u \alpha^* = \alpha^* \varphi_v$, for each 2-morphism $\begin{tikzcd} I \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & J \end{tikzcd}$.

The *composition* of two morphisms of prederivators $F : \mathbb{D} \to \mathbb{E}$ and $G : \mathbb{E} \to \mathbb{F}$ is defined by setting $(GF)_I = G_I F_I$, for all I in $\mathscr{D}ia$, and $(\gamma\varphi)_u = (G_J \varphi_u)(\gamma_u F_I)$, for all 1-morphisms $u : I \to J$ in $\mathscr{D}ia$.

If F and G are two morphisms of prederivators from $\mathbb{D}$ to $\mathbb{E}$, a 2-*morphism* $\alpha : F \to G$ is given by a functor $\alpha_I : F_I \to G_I$, for each I in $\mathscr{D}ia$, such that $(u^* \alpha_J)\varphi_u = \gamma_u(\alpha_I u^*)$, for all 1-morphisms $u : I \to J$ of $\mathscr{D}ia$. The *composition* of two such morphisms is clear. Thus, the 1-morphisms between any fixed pair of prederivators $\mathbb{D}$ and $\mathbb{E}$, and their 2-morphisms, are the objects and, respectively, the morphisms of a 1-category that we denote $\underline{\mathcal{H}om}(\mathbb{D}, \mathbb{E})$.

A *morphism of derivators* is just a morphism of the underlying prederivators. Thus, given a pair of derivators $\mathbb{D}$ and $\mathbb{E}$, we use the same notation $\underline{\mathcal{H}om}(\mathbb{D}, \mathbb{E})$ to indicate the 1-category of morphisms of derivators from $\mathbb{D}$ into $\mathbb{E}$.

Let us remark that, in the presence of another morphism $v : K \to I$, for any object X lying in $\mathbb{S}(J)$, the isomorphism φ_{uv}^X is explicitely given by the composition

$$F_K v^* u^* X \xrightarrow[\sim]{\varphi_v^{u^* X}} v^* F_I u^* X \xrightarrow[\sim]{v^*(\varphi_u^X)} v^* u^* F_J X.$$

The morphism φ_u induces, via the adjunction morphisms $\eta_u : \mathbf{1} \to u^* u_!$ and $\varepsilon_u : u_! u^* \to \mathbf{1}$, another morphism

$$\varphi^u : u_! F_I \to F_J u_! \ ,$$

whose action on any object X of $\mathbb{S}(I)$ is given by the composition

$$u_! F_I X \xrightarrow{u_! F_I(\eta_u^X)} u_! F_I u^* u_! X \xrightarrow[\sim]{u_!(\varphi_u^{u_! X})} u_! u^* F_J u_! X \xrightarrow{\varepsilon_u^{F_J u_! X}} F_J u_! X.$$

Analogously to the contravariant case, we have the relation

$$\varphi^{u, v_! X} \circ u_!(\varphi^{v,X}) = \varphi^{uv,X} \ ,$$

valid for all objects X of $\mathbb{S}(K)$. Indeed, functoriality of the morphisms involved and the relation $\varphi_{uv}^X = v^*(\varphi_u^X) \circ \varphi_v^{u^* X}$ give us commutative diagrams that we can use in the following

sequence of equalities

$$\begin{aligned}
\varphi^{uv,X} &= \varepsilon_{uv}^{F_J(uv)_!X} \circ (uv)_![\varphi_{uv}^{(uv)_!X}] \circ (uv)_!F_J(\eta_{uv}^X) \\
&= \varepsilon_{u}^{F_J(uv)_!X} \circ u_![\varepsilon_{v}^{u^*F_J(uv)_!X}] \circ (uv)_![v^*(\varphi_{u}^{(uv)_!X}) \circ \varphi_{v}^{u^*(uv)_!X}] \circ \\
&\quad \circ (uv)_!F_Jv^*(\eta_u^{v_!X}) \circ (uv)_!F_J(\eta_v^X) \\
&= \varepsilon_{u}^{F_J(uv)_!X} \circ u_![\varphi_{u}^{(uv)_!X}] \circ u_![\varepsilon_{v}^{F_Iu^*(uv)_!X}] \circ (uv)_!v^*F_I(\eta_u^{v_!X}) \circ \\
&\quad \circ (uv)_![\varphi_v^{v_!X}] \circ (uv)_!F_J(\eta_v^X) \\
&= \varepsilon_{u}^{F_J(uv)_!X} \circ u_![\varphi_{u}^{(uv)_!X}] \circ u_!F_I(\eta_u^{v_!X}) \circ u_![\varepsilon_v^{F_Iv_!X}] \circ u_!v_![\varphi_v^{v_!X}] \circ u_!v_!F_J(\eta_v^X) \\
&= \varphi^{u,v_!X} \circ u_!(\varphi^{v,X}).
\end{aligned}$$

2.3. **Exact categories and their derived categories.** In this subsection we recall the notion of exact category and the construction of the related derived category.

Definition 2.8. An *exact category* is an additive category $\mathcal{A}$ endowed with a class of *short exact sequences*, also said *exact pairs*,

$$X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z$$

(*i.e.*, $\mathsf{ker}(p) = i$, $\mathsf{cok}(i) = p$), which is closed under isomorphisms. Elements in this class are called *conflations* and the arrows of type $\rightarrowtail$ (resp., $\twoheadrightarrow$) are called *inflations* (resp., *deflations*), such that the following axioms hold.

Ex 0 The identity morphism $0 \longrightarrow 0$ is a deflation.

Ex 1 The class of deflations is stable under composition.

Ex 2 For any deflation $Y \overset{d}{\twoheadrightarrow} Z$ and any morphism $Z' \xrightarrow{f} Z$ in $\mathcal{A}$, there exists a cartesian square (pull-back)

$$\begin{tikzcd}
Y' \arrow[r, dashed, two heads, "d'"] \arrow[d, dashed, "f'"'] & Z' \arrow[d, "f"] \\
Y \arrow[r, two heads, "d"'] & Z
\end{tikzcd},$$

where f' is an arrow in $\mathcal{A}$ and d' is a deflation.

Ex2° For any inflation $X \overset{i}{\rightarrowtail} Y$ and any morphism $X \xrightarrow{f} X'$ in $\mathcal{A}$, there exists a cocartesian square (push-out)

$$\begin{tikzcd}
X \arrow[r, tail, "i"] \arrow[d, "f"'] & Y \arrow[d, dashed, "f'"] \\
X' \arrow[r, dashed, tail, "i'"'] & Y'
\end{tikzcd},$$

where f' is an arrow in $\mathcal{A}$ and i' is an inflation.

Example 2.9. The following are examples of exact categories.

a) The opposite category $\mathcal{A}^\circ$ of any exact category $\mathcal{A}$ canonically inherits a structure of exact category.
b) Any additive category $\mathcal{A}$, endowed with all of its split short exact sequences.

c) Let $\mathcal{A}$ be an exact category. Then, for any small category I the category defined by
$$\underline{\mathcal{A}}(I) := \underline{\mathsf{Hom}}(I^\circ, \mathcal{A})$$
becomes exact when endowed with componentwise conflations.

Remark that the exact category $\underline{\mathcal{A}}(I)$ usually have non split conflations even if all conflations of $\mathcal{A}$ split.

Let us call $\mathcal{C}^b\mathcal{A}$ the *category of bounded complexes*
$$\ldots \longrightarrow M^p \xrightarrow{d^p} M^{p+1} \longrightarrow \ldots , \qquad d^{p+1}d^p = 0 , \quad p \in \mathbb{Z} ,$$
over an exact category $\mathcal{A}$, where $M^p = 0$ for all $|p| >> 0$. The category $\mathcal{H}^b\mathcal{A}$ is the quotient of $\mathcal{C}^b\mathcal{A}$ with the ideal of nullhomotopic morphisms. Note that the category $\mathcal{H}^b\mathcal{A}$ is canonically triangulated, with suspension functor Σ given by
$$(\Sigma X)^p := X^{p+1} \quad , \quad d_{\Sigma X} := -d_X ,$$
and triangles obtained from the componentwise split short exact sequences.

A complex N is *strictly acyclic* if there exist conflations
$$0 \longrightarrow Z^p \rightarrowtail^{i^p} N^p \overset{q^p}{\twoheadrightarrow} Z^{p+1} \longrightarrow 0 \ , \ p \in \mathbb{Z} ,$$
such that $d^p = i^{p+1}q^p$, for all $p \in \mathbb{Z}$. A complex N is *acyclic* if it is isomorphic in $\mathcal{H}^b\mathcal{A}$ to a strictly acyclic complex. One can show that, if the category $\mathcal{A}$ is idempotent complete, then any complex is acyclic iff it is strictly acyclic. Moreover, (without the idempotent completeness condition) it is true that the images in $\mathcal{H}^b\mathcal{A}$ of the acyclic complexes form a thick subcategory $\mathcal{N}$ of $\mathcal{H}^b\mathcal{A}$, *i.e.*, $\mathcal{N}$ is a triangulated subcategory of $\mathcal{H}^b\mathcal{A}$ which is stable under retracts. The *(bounded) derived category* of $\mathcal{A}$, that we denote $\mathcal{D}^b\mathcal{A}$, is the Verdier quotient [16] $\mathcal{H}^b\mathcal{A}/\mathcal{N}$.

Note that there is a canonical embedding
$$can : \mathcal{A} \longrightarrow \mathcal{D}^b\mathcal{A} , \quad X \mapsto (\ldots \to 0 \to X \to 0 \to \ldots) , \quad \deg(X) = 0 \quad .$$
For any conflation in $\mathcal{A}$,
$$\varepsilon : \quad 0 \longrightarrow X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z \longrightarrow 0 \ ,$$
we have a canonical distinguished triangle in $\mathcal{D}^b\mathcal{A}$,
$$can(\varepsilon) : \quad can(X) \overset{can(i)}{\rightarrowtail} can(Y) \overset{can(p)}{\twoheadrightarrow} can(Z) \xrightarrow{\partial\varepsilon} \Sigma can(X) \ .$$
In this way, we have constructed a 2-functor
$$\mathscr{E}xa \to \mathscr{T}ria \quad , \quad \mathcal{A} \mapsto \mathcal{D}^b\mathcal{A} ,$$
from the 2-category of exact categories into the 2-category of triangulated categories.

The reason we consider derivators or other structures like towers is that the canonical embedding *can* does not have the universal extension property we are looking for.

2.4. **The triangulated derivator associated with an exact category.** Let $\mathcal{A}$ be an exact category. Here is one central notion of this paper.

Definition 2.10. $\mathbb{D}_\mathcal{A} : \mathscr{D}ia^\circ \to \mathscr{C}at$ is the prederivator that associates the bounded derived category $\mathcal{D}^b(\underline{\mathsf{Hom}}(I^\circ, \mathcal{A}))$ with any I in $\mathscr{D}ia$.

In the Appendix [10] to the article [13] B. Keller proves the following

Theorem 2.11. *Let us consider the restriction $\mathbb{D}_\mathcal{A}$ of the derivator in the definition 2.10 to the 2-subcategory $\mathscr{D}ia_\mathsf{f}$. Then, the prederivator $\mathbb{D}_\mathcal{A}$ is a triangulated derivator.*

Remark that triangularity is a property of a derivator as opposed to the triangulated extra structure that one can put on an additive category.

Let us denote $\underline{\mathbb{A}} : \mathscr{D}ia \to \mathscr{C}at$ the prederivator that associates the category $\underline{\mathbb{A}}(I) := \mathsf{Hom}(I^\circ, \mathcal{A})$ of contravariant functors from I into an exact category $\mathcal{A}$, with any diagram I in $\mathscr{D}ia$. It is easy to see that $\underline{\mathbb{A}}$ inherits from $\mathcal{A}$ the structure of an *exact derivator*, *i.e.*, the image of $\underline{\mathbb{A}}$ is in the 2-subcategory $\mathscr{E}xa$ of $\mathscr{C}at$ consisting of the small exact categories, if we let any $\underline{\mathbb{A}}(I)$ be endowed with short exact sequences which are argumentwise conflations.

Clearly, the canonical embedding of an exact category into its bounded derived category gives rise to a canonical morphism of prederivators, which we still call

$$can : \underline{\mathbb{A}} \longrightarrow \mathbb{D}_{\mathcal{A}} \,,$$

given, for any diagram I, by the canonical embedding

$$can_I : \underline{\mathbb{A}}(I) \longrightarrow \mathbb{D}_{\mathcal{A}}(I) = \mathcal{D}^b(\underline{\mathbb{A}}(I)) \ .$$

For each bicartesian square in $\underline{\mathbb{A}}(\square)$

(1)
$$X \quad = \quad \begin{tikzcd} X_{00} \arrow[r, tail] \arrow[d, two heads] & X_{01} \arrow[d, two heads] \\ X_{10} \arrow[r, tail] & X_{11} \end{tikzcd} \,,$$

its image $can_\square(X)$ is bicartesian in $\mathcal{D}^b(\underline{\mathbb{A}}(\square))$, *i.e.*, isomorphisms

$$X_{00} \xrightarrow{\sim} X_{10} \overset{\mathbf{R}}{\underset{X_{11}}{\prod}} X_{01} \qquad \text{and} \qquad X_{11} \xrightarrow{\sim} X_{10} \overset{\mathbf{L}}{\underset{X_{00}}{\coprod}} X_{01}$$

hold. We say that the morphism $can : \underline{\mathbb{A}} \to \mathbb{D}_{\mathcal{A}}$ is *exact*. More generally we have the following

Definition 2.12. Let $\mathbb{E}$ and $\mathbb{F}$ be triangulated derivators.

a) A morphism $F : \underline{\mathbb{A}} \to \mathbb{E}$ is said *exact* or *∂-morphism* if $F_\square(X)$ is bicartesian in $\mathbb{E}(\square)$ for each X as in (1).

b) A morphism of derivators $F : \mathbb{E} \to \mathbb{F}$ is *triangulated* if $F_\square(X) \in \mathbb{F}(\square)$ is bicartesian for each bicartesian $X \in \mathbb{E}(\square)$.

c) A morphism $\alpha : F \to G$ of ∂-morphisms of derivators F and G from $\underline{\mathbb{A}}$ to $\mathbb{E}$ is *exact* (or ∂) if, for each bicartesian square X in $\underline{\mathbb{A}}(\square)$ as in (1), the morphism

$$\alpha^X_\square : F_\square X \to G_\square X$$

is a *bicartesian morphism* of bicartesian squares in $\mathbb{E}(\square)$. More explicitly this means that the canonical morphisms

$$\eta^{\alpha^X_\square} : \alpha^X_\square \longrightarrow i_{\lrcorner *} i_\lrcorner{}^* \alpha^X_\square \,,$$

$$\varepsilon^{\alpha^X_\square} : i_{\ulcorner !} i_\ulcorner{}^* \alpha^X_\square \longrightarrow \alpha^X_\square \,,$$

induced by the adjunction isomorphisms $\eta^X : X \xrightarrow{\sim} i_{\lrcorner *} i_\lrcorner{}^* X$ and $\varepsilon^X : i_{\ulcorner !} i_\ulcorner{}^* X \xrightarrow{\sim} X$, are invertible.

d) A morphism $\alpha : F \to G$ of triangulated morphisms of derivators F and G from $\mathbb{E}$ to $\mathbb{F}$ is *triangulated* if, for each bicartesian square X in $\mathbb{E}(\square)$, the morphism

$$\alpha^X_\square : F_\square X \to G_\square X$$

is a *bicartesian morphism* of bicartesian squares in $\mathbb{F}(\square)$.

The *composition* of two such morphisms is clear. Thus, the ∂-morphisms between any fixed pair of derivators $\mathbb{A}$ and $\mathbb{E}$, and their ∂-morphisms, are the objects and, respectively, the morphisms of a 1-category that we indifferently denote $\underline{\mathcal{H}om}_{ex}(\mathbb{A},\mathbb{E})$ or $\underline{\mathcal{H}om}_{\partial}(\mathbb{A},\mathbb{E})$. Analogously, the category $\underline{\mathcal{H}om}_{tr}(\mathbb{E},\mathbb{F})$ is the 1-category of triangulated morphisms of derivators from $\mathbb{E}$ into $\mathbb{F}$ and their triangulated morphisms.

The following is the main result of this article.

Theorem 2.13. *Let $\mathbb{E}$ be a triangulated derivator of type $\mathscr{D}ia_{\mathsf{f}}$. Then, the canonical morphism $can : \underline{\mathbb{A}} \to \mathbb{D}_{\mathcal{A}}$ induces an equivalence of categories*

$$\underline{\mathcal{H}om}_{tr}(\mathbb{D}_{\mathcal{A}},\mathbb{E}) \xrightarrow{\sim} \underline{\mathcal{H}om}_{ex}(\underline{\mathbb{A}},\mathbb{E}).$$

Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}$ an exact category. A *∂-functor (also called exact functor)* is an additive functor $F : \mathcal{A} \to \mathcal{T}$ endowed with functorial distinguished triangles

$$F(X) \longrightarrow F(Y) \longrightarrow F(Z) \xrightarrow{\partial\varepsilon} \Sigma F(X)$$

for each conflation

$$\varepsilon : \; X \rightarrowtail Y \twoheadrightarrow Z \quad .$$

A *morphism of ∂-functors* $\alpha : F \to F'$ is an (additive) functor such that the square

$$\begin{tikzcd} FZ \arrow[r, "\partial\varepsilon"] \arrow[d, "\alpha^Z"'] & \Sigma FX \arrow[d, "\Sigma\alpha^X"] \\ F'Z \arrow[r, "\partial'\varepsilon"] & \Sigma F'Z \end{tikzcd}$$

commutes for each inflation ε. This also gives a 1-category that we indifferently denote $\underline{\mathsf{Hom}}_{ex}(\mathcal{A},\mathcal{T})$ or $\underline{\mathsf{Hom}}_{\partial}(\mathcal{A},\mathcal{T})$.

As a corollary of Theorem 2.13, we have the following

Theorem 2.14.

a) *Let $F : \mathcal{A} \to \mathcal{T}$ be a ∂-functor. Suppose that $\mathcal{T} = \mathbb{T}(e)$ for some triangulated derivator $\mathbb{T}$ of type $\mathscr{D}ia_{\mathsf{f}}$ and that*

$$\mathsf{Hom}_{\mathcal{T}}(F(X), \Sigma^n F(Y)) = 0$$

for each $n < 0$, for all X, Y in $\mathcal{A}$ (Toda condition). Then, the functor F extends uniquely (up to a unique basic morphism) to a basic triangulated functor

$$\widetilde{F} : \mathcal{D}^b\mathcal{A} \to \mathcal{T}.$$

b) *Let $F, F' : \mathcal{D}^b\mathcal{A} \to \mathcal{T}$ be two basic triangulated functors such that the (Toda) conditions*

$$\begin{aligned} \mathsf{Hom}_{\mathcal{T}}(\Sigma^n F(X), F(Y)) = 0\,, &\qquad n > 0\,,\\ \mathsf{Hom}_{\mathcal{T}}(\Sigma^n F'(X), F'(Y)) = 0\,, &\qquad n > 0\,,\\ \mathsf{Hom}_{\mathcal{T}}(\Sigma^n F(X), F'(Y)) = 0\,, &\qquad n > 0\,, \end{aligned}$$

hold for all X, Y in $\mathcal{A}$. Suppose that μ is a morphism between the restrictions of F and F' to $\mathcal{A}$. Then, the morphism μ extends uniquely (up to a unique basic morphism of morphisms of derivators) to a basic morphism of triangulated functors

$$\widetilde{\mu} : F \to F'.$$

Remark 2.15. The extension of $F : \mathcal{A} \to \mathcal{T}$ to the derived category is unique once the derivator $\mathbb{T}$ is fixed. The question whether the extension is not unique since there might be more than one triangulated derivator with the same base $\mathcal{T}$ remains open.

Let us notice that item b) in Theorem 2.14 *differs* from the analogous fact, true for the towers, which is contained in Corollary 2.7 in [9]. In fact, the same conclusion there follows from the last of our three Toda conditions alone. The analogous situation holds for the item b) in our Theorem 6.5 when compared with Keller's Theorem 2.7 in [9].

Let us recall that Heller, Franke, Cisinski, ... have shown universality properties of many important derivators, *e.g.* ,

Theorem 2.16 (Heller [7], Franke [4]). *Let $\mathbb{S}p : \mathscr{D}ia_{\mathsf{f}} \to \mathscr{C}at$ be the triangulated derivator associated with the homotopy category of finite spectra,* i.e.*, spectra with finitely many cells (more precisely with a localizer defining this homotopy category). Let $\mathbb{D} : \mathscr{D}ia_{\mathsf{f}} \to \mathscr{C}at$ be a triangulated derivator. Then, we have the isomorphism of categories*

$$\underline{\mathcal{H}om}_!(\mathbb{S}p, \mathbb{D}) \xrightarrow{\sim} \mathbb{D}(e), \quad F \mapsto (F_e)(S^0).$$

Here ! indicates commutativity with all Kan extensions.

3. Keller's towers

We begin by defining a 2-category that we call $\mathscr{C}ubes$. For any positive integer n, let us denote C_n the n-product $\Delta_1 \times \Delta_1 \times \ldots \times \Delta_1$. Clearly, C_0 is the one object category $e = \{0\}$ and C_2 is the diagram $\square$. Sometimes, we will refer to these diagrams as n-cubes. These diagrams are the objects of the 2-category $\mathscr{C}ubes$.

Let us think of the objects C_n as the partially ordered n-dimensional cubes with side of length one, so that any vertex x is identified by an n-tuple $(x_1, \ldots, x_n)$ of numbers in the set $\{0, 1\}$. The 1-morphisms in $\mathscr{C}ubes$ are all possible compositions of the following order preserving maps. For any $n \in \mathbb{N}$,

$$i^j_\varepsilon : C_n \longrightarrow C_{n+1}, \qquad (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{j-1}, \varepsilon, x_j, \ldots, x_n)\,,$$
$$p^j : C_{n+1} \longrightarrow C_n, \qquad (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)\,,$$

where $\varepsilon \in \{0, 1\}$. Clearly, there are relations among these functors (see [9]). The 2-morphisms in $\mathscr{C}ubes$ are given by the following order relation : for any pair of 1-morphisms u, v from C_l to C_m, we write $u \Rightarrow v$ if there is an arrow $u(x) \to v(x)$, for all $x \in C_l$.

Notice that $\mathscr{C}ubes$ is *not* a full 2-subcategory of $\mathscr{D}ia_{\mathsf{f}}$.

Definition 3.1. A *Keller's tower of additive categories* or simply a *tower of additive categories* E is a 2-functor from the opposite 2-category $Cubes^\circ$ to the 2-category of additive categories $\mathscr{A}dd$.

For any 1-morphism $u : C_l \to C_m$ and any 2-morphism $\alpha : u \Rightarrow v$ in $Cubes$, we denote $u^* : \mathsf{E}(C_m) \to \mathsf{E}(C_l)$ the induced 1-morphism and $\alpha^* : v^* \Rightarrow u^*$ the induced 2-morphism, as in the case of derivators.

Clearly, any additive (pre-)derivator $\mathbb{E}$ gives rise to a tower E when it is restricted to $\mathscr{C}ubes^\circ$. An important example of additive tower is the restriction of the prederivator $\underline{\mathbb{A}}$ defined in 2.4 to $\mathscr{C}ubes^\circ$.

If the category $\mathcal{A}$ has an exact structure, then we can speak of a *tower of exact categories* $\underline{\mathsf{A}}$ by endowing each $\underline{\mathsf{A}}(C_n)$ with the pairs of composable morphisms of functors (*i.e.*, natural transformations) whose evaluation at each $x \in C_n^\circ$ is a conflation of $\mathcal{A}$. Analogously, we can define the notion of *tower of triangulated categories* as a contravariant 2-functor from $\mathscr{C}ubes$ to the 2-category of triangulated categories $\mathscr{T}ria$. As an important example we can consider, for any arbitrary exact category $\mathcal{A}$, the triangulated tower defined by $\mathsf{D}_\mathcal{A} : \mathscr{C}ubes^\circ \to \mathscr{C}at$, *i.e.*, the tower that associates the bounded derived category $\mathcal{D}^b(\underline{\mathsf{Hom}}(C_n^\circ, \mathcal{A}))$ with any C_n in $\mathscr{C}ubes$. With analogy to the derivator case, we can define the *morphisms of towers* (called 'towers of morphisms' in [9]) and their compositions.

Definition 3.2. Let D and E be two towers. A *morphism of towers* $F : \mathsf{D} \to \mathsf{E}$ consists of

- an additive functor $F_n : \mathsf{D}_n \to \mathsf{E}_n$, for each $n \in \mathbb{N}$;
- an isomorphism of functors

$$\varphi_u : F_m u^* \xrightarrow{\sim} u^* F_n \ ,$$

for each 1-morphism $u : C_m \to C_n$ in $\mathscr{C}ubes$;

such that

- $\varphi_{\mathbf{1}_n} = \mathbf{1}_{F_n}$, for each $n \in \mathbb{N}$;
- $\varphi_{uv} = (v^* \varphi_u)(\varphi_v u^*)$, for each pair of 1-morphisms $C_l \xrightarrow{v} C_m \xrightarrow{u} C_n$ in $\mathscr{C}ubes$;
- $\varphi_u \alpha^* = \alpha^* \varphi_v$, for each 2-morphism $\begin{tikzcd} C_m \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & C_n \end{tikzcd}$ in $\mathscr{C}ubes$.

The *composition* of two morphisms of towers $F : \mathsf{D} \to \mathsf{E}$ and $G : \mathsf{E} \to \mathsf{F}$ is defined by setting $(GF)_n = G_n F_n$, for all $n \in \mathbb{N}$, and $(\gamma\varphi)_u = (G_n \varphi_u)(\gamma_u F_m)$, for all 1-morphisms $u : C_m \to C_n$ in $\mathscr{C}ubes$.

If F and G are two morphisms of towers from D to E, a 2-*morphism* $\alpha : F \to G$ is given by a functor $\alpha_n : F_n \to G_n$, for each n in $\mathbb{N}$, such that $(u^*\alpha_n)\varphi_u = \gamma_u(\alpha_m u^*)$, for all 1-morphisms $u : C_m \to C_n$ of $\mathscr{C}ubes$. The *composition* of two such morphisms is clear. Thus, the 1-morphisms between any fixed pair of towers D and E and the 2-morphisms among them are the objects and, respectively, the morphisms of an additive 1-category that we denote $\underline{\mathcal{H}om}_{add}(\mathsf{D}, \mathsf{E})$, or simply $\underline{\mathcal{H}om}(\mathsf{D}, \mathsf{E})$.

It is clear how to define morphisms of exact towers (resp., morphisms of triangulated towers) and morphisms among them. Thus, for any pair of exact (resp., triangulated) towers D and E, we get the 1-category $\underline{\mathcal{H}om}_{ex}(\mathsf{D}, \mathsf{E})$ (resp., $\underline{\mathcal{H}om}_{tr}(\mathsf{D}, \mathsf{E})$). A morphism of towers $F : \mathsf{E} \to \mathsf{F}$ from a tower of exact categories E into a tower of triangulated categories F is a ∂-*morphism* or *exact morphism of towers* from E to F if $F_n : \mathsf{E}_n \to \mathsf{F}_n$, $n \in \mathbb{N}$, is a sequence of ∂-functors. We denote $\underline{\mathcal{H}om}_{ex}(\mathsf{E}, \mathsf{F})$ the 1-category of these functors.

Theorem 3.3 (Keller, [9])**.** *Let $\mathcal{A}$ be an exact category and E a tower of triangulated categories. Then, the canonical morphism $can : \underline{\mathsf{A}} \to \mathsf{D}_{\mathcal{A}}$ induces an equivalence of categories*

$$\underline{\mathcal{H}om}_{tr}(\mathsf{D}_{\mathcal{A}}, \mathsf{E}) \xrightarrow{\sim} \underline{\mathcal{H}om}_{ex}(\underline{\mathsf{A}}, \mathsf{E}).$$

Clearly, this is an important theorem. The aim of the present article is to show that it also holds in the context of derivators.

4. Epivalence and Recollement

In this section we want to show a very important property of triangulated derivators and their morphisms. We will see that the property of recollement, which is enjoyed by derivators by the very definition, is important in this setting.

Definition 4.1. A *recollement* of triangulated categories $\mathcal{T}'$, $\mathcal{T}$ and $\mathcal{T}''$ is a diagram of triangulated functors

$$\begin{tikzcd} \mathcal{T}' \arrow[r, "j_!"] & \mathcal{T} \arrow[l, bend right=50, "j^?"'] \arrow[l, bend left=50, "j^*"] \arrow[r, "i^*"] & \mathcal{T}'' \arrow[l, bend right=50, "i_!"'] \arrow[l, bend left=50, "i_*"] \end{tikzcd}$$

such that

- the pairs $j^? \dashv j_!$, $j_! \dashv j^*$, $i_! \dashv i^*$, $i^* \dashv i_*$ are adjunctions;
- $i^* j_! = 0$;

- the functors i_*, $i_!$ and $j_!$ are fully faithful;
- for every object X lying in $\mathcal{T}$, there are distinguished triangles

$$i_!i^*X \xrightarrow{\varepsilon_i^X} X \xrightarrow{\eta_j^X} j_!j^?X \longrightarrow \Sigma i_!i^*X\ ,$$

$$j_!j^*X \xrightarrow{\varepsilon_j^X} X \xrightarrow{\eta_i^X} i_*i^*X \longrightarrow \Sigma j_!j^*X\ ,$$

where $(\varepsilon_i^X, \eta_i^X)$ and $(\varepsilon_j^X, \eta_j^X)$ are pairs of adjunction morphisms related to i and j, respectively.

The functors $j^?$, $j_!$, j^*, $i_!$, i^*, i_* are also known as the 6-*gluing functors.*

Let us recall that in the context of derivators open / closed inclusions of diagrams naturally give rise to recollements of triangulated categories. This allows us constructing the shift and loop autoequivalences in terms of the 6-gluing functors as in the following lemma that we state even if we are not going to use it in what follows.

Lemma 4.2. *Let $\mathbb{S}$ be a triangulated derivator. Fix an arbitrary diagram I in $\mathscr{D}ia$, let $j : I \to I \times \Delta_1$ and $i : I \to I \times \Delta_1$ be the obvious open and closed inclusions, respectively. Then, there are* canonical *isomorphisms of functors*

$$j^?i_* \tilde{\to} \Sigma_I \quad , \quad \Omega_I = \Sigma_I^{-1} \tilde{\to} i^!j_!\ .$$

Proof. It is well known (*e.g.* , look at [3, Section 9]) that an open inclusion j with a closed inclusion i such that the diagram spanned by the union of their images is all of $I \times \Delta_1$ give rise to a recollement of triangulated categories

$$\begin{tikzcd} \mathbb{S}(I) \arrow[r, "j_!"] & \mathbb{S}(I \times \Delta_1) \arrow[l, bend right, "j^?"'] \arrow[l, bend left, "j^*"] \arrow[r, "i^*"] & \mathbb{S}(I). \arrow[l, bend right, "i_!"'] \arrow[l, bend left, "i_*"] \end{tikzcd}$$

Thus, for any object X in $\mathbb{S}(I \times \Delta_1)$ there is a distinguished triangle

$$i_!i^*X \longrightarrow X \longrightarrow j_!j^?X \longrightarrow \Sigma_{I\times\Delta_1} i_!i^*X.$$

Here, by applying the functor $\mathsf{Hom}(-, j_!j^?X)$ and passing to long exact sequence, we see that the connecting morphism $j_!j^?X \to \Sigma i_!i^*X$ is *unique* because $\mathsf{Hom}(\Sigma i_!i^*X, j_!j^?X) = 0$ by adjunction and by recollement.

Let us apply the triangulated functor j^* to this triangle. Because of the recollement axioms we get a distinguished triangle in $\mathbb{S}(I)$

$$i^*X \longrightarrow j^*X \longrightarrow j^?X \longrightarrow \Sigma_I i^*X.$$

In particular, when the object X is of the form i_*Y, for some object Y in $\mathbb{S}(I)$, the triangle becomes

$$i^*i_*Y \longrightarrow j^*i_*Y \longrightarrow j^?i_*Y \longrightarrow \Sigma_I i^*i_*Y.$$

Here $i^*i_*Y = Y$ because the functor i_* is fully faithful and $j^*i_*Y = 0$ by adjunction. It survives a functorial iso $j^?i_*Y \tilde{\to} \Sigma_I Y$, for all Y in $\mathbb{S}(I)$. The conclusion follows easily.

The analogous proof of the second isomorphism starts by applying the triangulated functor $i^!$ to the canonical distinguished triangle

$$\Sigma^{-1}i_*i^*X \longrightarrow j_!j^*X \xrightarrow{\varepsilon_j^X} X \xrightarrow{\eta_i^X} i_*i^*X$$

and taking $X = j_*Y$. □

If we are willing to study commutativity of derivator morphisms with the 6-gluing functors we need another concept.

Definition 4.3. Let $\mathcal{A}$ be an exact category and let $\mathcal{S}$, $\mathcal{T}$ be triangulated categories. We say that

a) an additive functor $F : \mathcal{S} \to \mathcal{T}$ is *weakly triangulated* if, for each distinguished triangle
$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$
of $\mathcal{S}$, there is *some* distinguished triangle
$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{w'} \Sigma F(X)$$
of $\mathcal{T}$;

b) an additive functor $F : \mathcal{A} \to \mathcal{S}$ is a *weak ∂-functor* or *weakly exact* if, for each conflation
$$X \rightarrowtail^{u} Y \xrightarrow{v}\!\!\!\!\rightarrow Z$$
of $\mathcal{A}$, there is *some* distinguished triangle
$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{z} \Sigma F(X)$$
of $\mathcal{S}$.

Clearly, the notion of weakly triangulated functor is much weaker than the related notion of triangulated functor, which is described in [12] under the name of S-functor. Nevertheless we will see that, when the triangulated categories we are considering arise in the context of triangulated derivators, this notion of weakly triangulated functor canonically extends to the usual notion of triangulated functor. In other words, it is redundant to ask that all the functors F_I are triangulated in order they extend to a morphism of triangulated derivators. Even less is required for this happens, it is enough to ask that F_{Δ_1} and $F_{\square}$ are *weakly* triangulated functor.

Let $\mathbb{S}$ and $\mathbb{T}$ be triangulated derivators and $\underline{\mathbb{A}}$ an exact derivator. Let $F : \mathbb{S} \to \mathbb{T}$ and $F : \underline{\mathbb{A}} \to \mathbb{S}$ be *additive* functors of derivators. Let us fix an arbitrary diagram I in $\mathscr{D}ia$. Let $j : I \to I \times \Delta_1$ and $i : I \to I \times \Delta_1$ be the obvious open and closed inclusions, respectively. We remark that in this situation the natural transformation (*cf.* subsection 2.2) $\varphi^j : j_! F_I \to F_{I\times\Delta_1} j_!$ is invertible. Indeed, since every point inclusion $e \to I \times \Delta_1$ either factors through i or j, it follows that i^* and j^* detect isomorphisms and we can easily check that $i^*(\varphi^{j,X}) = 0$ and $j^*(\varphi^{j,X})$ are isomorphisms, for all X in $\mathbb{S}(I \times \Delta_1)$.

Thus, there is an induced natural transformation
$$\psi_j : j^? F_{I\times\Delta_1} \to F_I j^? ,$$
whose action ψ_j^X on an object X is given by the composition
$$j^? F_{I\times\Delta_1} X \xrightarrow{j^? F_{I\times\Delta_1}(\eta_j^X)} j^? F_{I\times\Delta_1} j_! j^? X \xrightarrow[\sim]{j^?(\varphi^{j,j^? X})^{-1}} j^? j_! F_I j^? X \xrightarrow[\sim]{\varepsilon_j^{F_I j^? X}} F_I j^? X.$$
Also, there is an induced natural transformation
$$\psi_{jj} : j_! j^? F_{I\times\Delta_1} \to F_{I\times\Delta_1} j_! j^? \quad ,$$
whose action on every X in $\mathbb{S}(I \times \Delta_1)$ is defined by the composition
$$j_! j^? F_{I\times\Delta_1} X \xrightarrow{j_!(\psi_j^X)} j_! F_I j^? X \xrightarrow[\sim]{\varphi^{j,j^? X}} F_{I\times\Delta_1} j_! j^? X.$$

Lemma 4.4. *In the situation just described for the* additive *morphism $F : \mathbb{S} \to \mathbb{T}$ of triangulated derivators, suppose that the functor $F_{I\times\Delta_1}$ is* weakly *triangulated. Then, for any X in $\mathbb{S}(I \times \Delta_1)$, there is a* unique *isomorphism*

$$\psi_{jj}^X := \varphi^{j_! j^? X} \circ \psi_j^X : j_! j^? F_{I\times\Delta_1} X \tilde{\to} F_{I\times\Delta_1} j_! j^? X$$

such that the relation $\psi_{jj}^X \circ \eta_j^{F_{I\times\Delta_1} X} = F_{I\times\Delta_1}(\eta_j^X)$ holds. In particular, the morphism of functors

$$\psi_j^X : j^? F_{I\times\Delta_1} X \to F_I j^? X$$

is invertible, for all X in $\mathbb{S}(I \times \Delta_1)$. More generally, $\psi_{jj} : j_! j^? F_{I\times\Delta_1} \tilde{\to} F_{I\times\Delta_1} j_! j^?$ and $\psi_j : j^? F_{I\times\Delta_1} \tilde{\to} F_I j^?$ are isomorphisms of functors.

Proof. For any object X in $\mathbb{S}(I\times\Delta_1)$, let us start by considering the canonical distinguished triangle that we have just seen in the proof of Lemma 4.2

$$i_! i^* X \xrightarrow{\varepsilon_i^X} X \xrightarrow{\eta_j^X} j_! j^? X \xrightarrow{\partial^X} \Sigma i_! i^* X ,$$

whose connecting morphism $\partial^X : j_! j^* X \to \Sigma i_! i^* X$ is *unique* because $\mathsf{Hom}(\Sigma i_! i^* X, j_! j^* X) = 0$ by adjunction and by recollement.

Now apply the *weakly* triangulated functor $F_{I\times\Delta_1}$ and get a distinguished triangle in $\mathbb{T}(I \times \Delta_1)$

$$F_{I\times\Delta_1} i_! i^* X \xrightarrow{F(\varepsilon)} F_{I\times\Delta_1} X \xrightarrow{F(\eta)} F_{I\times\Delta_1} j_! j^? X \xrightarrow{w} \Sigma F_{I\times\Delta_1} i_! i^* X ,$$

where we shortly write $F(\varepsilon)$ for $F_{I\times\Delta_1}(\varepsilon_i^X)$, $F(\eta)$ for $F_{I\times\Delta_1}(\eta_j^X)$. Recall that here the morphism $w : F_{I\times\Delta_1} j_! j^? X \to \Sigma F_{I\times\Delta_1} i_! i^* X$ just exists, *i.e.*, it is not unique, nor canonically constructed.

Let us consider the object $F_{I\times\Delta_1} X$ and write the related distinguished triangle

$$i_! i^* F_{I\times\Delta_1} X \xrightarrow{\varepsilon^F} F_{I\times\Delta_1} X \xrightarrow{\eta^F} j_! j^? F_{I\times\Delta_1} X \xrightarrow{\partial^F} \Sigma i_! i^* F_{I\times\Delta_1} X.$$

Here we identify $\varepsilon^F = \varepsilon_i^{F_{I\times\Delta_1} X}$, $\eta^F = \eta_j^{F_{I\times\Delta_1} X}$ and $\partial^F = \partial^{F_{I\times\Delta_1} X}$ for the *unique* connecting morphism. Since there is a natural transformation (*cf.* subsection 2.2)

$$\varphi_{ii} := (\varphi^i i^*) \circ i_!(\varphi_i^{-1}) : i_! i^* F_{I\times\Delta_1} \tilde{\to} i_! F_I i^* \to F_{I\times\Delta_1} i_! i^* ,$$

for every object X in $\mathbb{S}(I \times \Delta_1)$ there are a morphism $\varphi_{ii}^X = ((\varphi^i i^*) \circ i_!(\varphi_i^{-1}))^X$ and an induced morphism of distinguished triangles of $\mathbb{T}(I \times \Delta_1)$

$$\begin{tikzcd}
i_! i^* F_{I\times\Delta_1} X \arrow[r, "\varepsilon^F"] \arrow[d, "\varphi_{ii}^X"] & F_{I\times\Delta_1} X \arrow[r, "\eta^F"] \arrow[d, equal, "\mathbf{1}"] & j_! j^? F_{I\times\Delta_1} X \arrow[r, "\partial^F"] \arrow[d, dashed, "\psi"] & \Sigma i_! i^* F_{I\times\Delta_1} X \arrow[d, "\Sigma\varphi_{ii}^X"] \\
F_{I\times\Delta_1} i_! i^* X \arrow[r, "F(\varepsilon)"] & F_{I\times\Delta_1} X \arrow[r, "F(\eta)"] & F_{I\times\Delta_1} j_! j^? X \arrow[r, "w"] & \Sigma F_{I\times\Delta_1} i_! i^* X.
\end{tikzcd}$$

Indeed, let us check commutativity of the square on the left. We have

$$\begin{aligned}
F_{I\times\Delta_1}(\varepsilon_i^X)\circ\varphi_{ii}^X &= F_{I\times\Delta_1}(\varepsilon_i^X)\circ\varepsilon_i^{F_{I\times\Delta_1}i_!i^*X}\circ i_!(\varphi_i^{i_!i^*X})\circ i_!F_I(\eta_i^{i^*X})\circ i_![(\varphi_i^X)^{-1}]\\
&= \varepsilon_i^{F_{I\times\Delta_1}X}\circ i_!i^*F_{I\times\Delta_1}(\varepsilon_i^X)\circ i_!(\varphi_i^{i_!i^*X})\circ i_!F_I(\eta_i^{i^*X})\circ i_![(\varphi_i^X)^{-1}]\\
&= \varepsilon_i^{F_{I\times\Delta_1}X}\circ i_!(\varphi_i^X)\circ i_!F_Ii^*(\varepsilon_i^X)\circ i_!F_I(\eta_i^{i^*X})\circ i_![(\varphi_i^X)^{-1}]\\
&= \varepsilon_i^{F_{I\times\Delta_1}X}\circ i_!(\varphi_i^X)\circ i_![(\varphi_i^X)^{-1}]\\
&= \varepsilon_i^{F_{I\times\Delta_1}X}.
\end{aligned}$$

Here, every canonical isomorphism comes from functoriality, as φ_i is an isomorphism of functors, and from the relation $i^*(\varepsilon_i^X)\circ\eta_i^{i^*X}=\mathbf{1}^{i^*X}$.

Let us remark the useful fact that, if we call p the obvious projection functor $I\times\Delta_1\to I$, then there are the adjunction morphisms $j\dashv p\dashv i$. This fact implies that $j^*\dashv p^*\dashv i^*$ also are adjunction morphisms. In particular, we get the equality $i_!=p^*$, which entails that the canonical morphisms $\varphi:=\varphi_{ii}^X=\varphi_{i\circ p}^X:(i\circ p)^*F_{I\times\Delta_1}X\to F_{I\times\Delta_1}(i\circ p)^*X$ and $\Sigma\varphi$ are invertible.

Thus, we have shown that ψ is an isomorphism. Moreover, it is unique because we can see that the group

$$\begin{aligned}
\mathsf{Hom}(\Sigma F_{I\times\Delta_1}i_!i^*X,j_!j^?F_{I\times\Delta_1}X) &= \mathsf{Hom}(\Sigma i_!i^*F_{I\times\Delta_1}X,j_!j^?F_{I\times\Delta_1}X)\\
&= \mathsf{Hom}(i_!i^*\Sigma F_{I\times\Delta_1}X,j_!j^?F_{I\times\Delta_1}X)\\
&= \mathsf{Hom}(i^*\Sigma F_{I\times\Delta_1}X,i^*j_!j^?F_{I\times\Delta_1}X)\\
&= 0
\end{aligned}$$

is canonically trivial via the canonical isomorphism φ, adjunction and the canonical isomorphism $i^*j_!=0$, thanks to the recollement axioms.

Therefore, if we are able to show that the morphism ψ_{jj}^X makes the central square of the diagram commutative too, we get $\psi=\psi_{jj}^X$ canonically and the statement follows. Indeed, there is a commutative diagram, by functoriality of η_j,

$$\begin{tikzcd}
j_!j^?F_{I\times\Delta_1}j_!j^?X & j_!j^?j_!F_Ij^?X \arrow[l, "j_!j^?(\varphi^{j,j^?X})"', "\sim"] \\
F_{I\times\Delta_1}j_!j^?X \arrow[u, "\eta_j^{F_{I\times\Delta_1}j_!j^?X}"] & j_!F_Ij^?X. \arrow[l, "\sim", "\varphi^{j,j^?X}"'] \arrow[u, "\sim"', "\eta_j^{j_!F_Ij^?X}"']
\end{tikzcd}$$

We already know that the three arrows labeled with the symbol ~ are isomorphisms. Therefore, the arrows $\eta_j^{F_{I\times\Delta_1}j_!j^?X}$ and $j_!(\varepsilon_j^{F_Ij^?X}):j_!j^?j_!F_Ij^?X\to j_!F_Ij^?X$ are invertible, too. In particular, we get that the composition $\varphi^{j,j^?X}\circ j_!(\varepsilon_j^{F_Ij^?X})\circ j_!j^?(\varphi^{j,j^?X})^{-1}$ gives us an inverse to $\eta_j^{F_{I\times\Delta_1}j_!j^?X}$. Now, since the relation $j_!j^?F_{I\times\Delta_1}(\eta_j^X)\circ\eta_j^{F_{I\times\Delta_1}X}=\eta_j^{F_{I\times\Delta_1}j_!j^?X}\circ F_{I\times\Delta_1}(\eta_j^X)$ holds, it follows that $\psi_{jj}^X\circ\eta_j^{F_{I\times\Delta_1}X}=F_{I\times\Delta_1}(\eta_j^X)$ also holds.

Since we have shown that ψ_{jj}^X is invertible, we get that the morphism $j_!(\psi_j^X)$ must be invertible, too. Now apply j^* to see that ψ_j^X is an isomorphism. The claim follows. □

We need a similar lemma for morphisms defined over *exact* derivators.

Lemma 4.5. *Let $\underline{\mathbb{A}}$ be the exact derivator induced by an exact category $\mathcal{A}$ and let $\mathbb{S}$ be a triangulated derivator. Let $F:\underline{\mathbb{A}}\to\mathbb{S}$ be an* additive *morphism of derivators such that the functor $F_{I\times\Delta_1}$ is* weakly *exact. Consider an arbitrary object X in $\underline{\mathbb{A}}(I\times\Delta_1)$ such that*

the 'vertical parallel arrows' of his diagram are deflations. More precisely, this means that $i_k^* i^* X \to i_k^* j^* X$ *is a deflation, for all* $k \in I$ *(here,* i *and* j *are as in lemma 4.4 and* $i_k : e \to I$ *is the obvious map).*

Then, there is a unique *isomorphism*

$$\psi_{jj}^X := \varphi^{j,j^? X} \circ \psi_j^X : j_! j^? F_{I\times\Delta_1} X \tilde{\to} F_{I\times\Delta_1} j_! j^? X$$

such that the relation $\psi_{jj}^X \circ \eta_j^{F_{I\times\Delta_1} X} = F_{I\times\Delta_1}(\eta_j^X)$ *holds. In particular, the morphism of functors*

$$\psi_j^X : j^? F_{I\times\Delta_1} X \to F_I j^? X$$

is invertible. More generally, when restricted on the subcategory of squares as above, $\psi_{jj} : j_! j^? F_{I\times\Delta_1} \tilde{\to} F_{I\times\Delta_1} j_! j^?$ *and* $\psi_j : j^? F_{I\times\Delta_1} \tilde{\to} F_I j^?$ *are isomorphisms of functors.*

Proof. The basic category $\mathcal{A} = \underline{\mathbb{A}}(e)$ is additive and has a zero object 0. By [5, Cor. 3.8] our derivator $\underline{\mathbb{A}}$ is *pointed* according to our definition 2.3, *i.e.*, the adjoint (exact) functors $i^!$ and $i^?$ of axiom **Der 6** exist. Notice that it is enough that a derivator is pointed in order to define the suspension and the loop endofunctors Σ_I and Ω_I, for all the diagrams I in $\mathscr{D}ia$ (*cf.* [5]).

Let us consider an arbitrary object X of $\underline{\mathbb{A}}(I \times \Delta_1)$ with the required property. The adjunction morphism

$$i_! i^* X \xrightarrow{\varepsilon_i^X} X$$

is a deflation. Indeed, it is clear that $i_k^* i^*(i_! i^* X) = i_k^* i^* X$, for all $k \in I$. Moreover, we can locally check that

$$i_k^* j^*(i_! i^* X) = i_k^* j^* p^* i^* X = (i\, p\, j\, i_k)^* X = i_k^* i^* X\ ,$$

for all $k \in I$. This means that the diagram of the object $i_! i^* X$ contains two identical horizontal subdiagrams, linked by vertical parallel identities. It follows that the arrow $i_k^* i^*(\varepsilon_i^X)$ is an identity, for all $k \in I$, and that $i_k^* j^*(\varepsilon_i^X)$ is the original deflation $i_k^* i^* X \to i_k^* j^* X$, for all $k \in I$.

Since the morphism ε_i^X is a deflation, it must fit into a conflation

$$\mathsf{ker}(\varepsilon_i^X) \rightarrowtail i_! i^* X \twoheadrightarrow X\ ,$$

whose image in $\mathbb{D}_{\mathcal{A}}(I \times \Delta_1)$ under the exact functor $can_{I\times\Delta_1}$ we already know to fit into a *canonical* distinguished triangle (*cf.* lemma 4.4)

$$\Sigma^{-1} j_! j^? X \longrightarrow i_! i^* X \longrightarrow X \longrightarrow j_! j^? X\ .$$

Uniqueness allows us canonically identifying the object $\mathsf{ker}(\varepsilon_i^X)$ with the loop object $\Omega j_! j^* X$ (*cf.* def. 3.19 in [5]).

Let us apply the *weakly* exact functor $F_{I\times\Delta_1}$ and shift to get a distinguished triangle in $\mathbb{S}(I \times \Delta_1)$

$$F_{I\times\Delta_1} i_! i^* X \longrightarrow F_{I\times\Delta_1} X \longrightarrow F_{I\times\Delta_1} j_! j^? X \longrightarrow \Sigma F_{I\times\Delta_1} i_! i^* X\ .$$

At this point we have reached the triangulated world and the proof goes on as in the proof of lemma 4.4. □

4.1. **The redundancy of the connecting morphism.** Let us recall that if $F : \mathbb{S} \to \mathbb{T}$ is a triangulated morphism of triangulated derivators then there are autoequivalences $\Sigma_I^{\mathbb{S}}$ and $\Sigma_I^{\mathbb{T}}$ at each diagram I such that, canonically, the induced functor $F_I : \mathbb{S}(I) \to \mathbb{T}(I)$ is a triangulated functor, for all diagrams I. This means that there is a canonical isomorphism of triangulated functors $\delta_I : F_I\Sigma_I^{\mathbb{S}} \tilde{\to} \Sigma_I^{\mathbb{T}} F_I$, for each diagram I. Altogheter, all these morphisms define a canonical 2-isomorphism $\delta : F\Sigma \tilde{\to} \Sigma F$.

Clearly, we may forget the property of a morphism of derivators of being triangulated. Consequently, we also forget the *canonical* morphism δ and think of the underlying additive structure of categories and morphisms only.

Proposition 4.6. *Let $\underline{\mathbb{A}}$ be the exact derivator defined by an exact category $\mathcal{A}$. Let $\mathbb{S}$ and $\mathbb{T}$ be triangulated derivators. Let us suppose that all these derivators be of some type $\mathscr{Dia}$.*

a) *The forgetful functor*

$$\underline{\mathcal{H}om}_{tr}(\mathbb{S}, \mathbb{T}) \to \underline{\mathcal{H}om}_{add}(\mathbb{S}|, \mathbb{T}|)$$

is an isomorphism onto the full subcategory consisting of the additive morphisms of derivators F such that $F_{\Delta_1} : \mathbb{S}(\Delta_1) \to \mathbb{T}(\Delta_1)$ and $F_\square : \mathbb{S}(\square) \to \mathbb{T}(\square)$ are weakly *triangulated functors with respect to the canonical triangulated structures of these categories.*

b) *The forgetful functor*

$$\underline{\mathcal{H}om}_{\partial}(\underline{\mathbb{A}}, \mathbb{S}) \to \underline{\mathcal{H}om}_{add}(\underline{\mathbb{A}}|, \mathbb{S}|)$$

is an isomorphism onto the full subcategory consisting of the additive morphisms of derivators F such that $F_{\Delta_1} : \underline{\mathbb{A}}(\Delta_1) \to \mathbb{S}(\Delta_1)$ and $F_\square : \underline{\mathbb{A}}(\square) \to \mathbb{S}(\square)$ are weak *∂-functors with respect to the canonical additive and triangulated structures of these categories.*

Proof. The proof of item a) (resp., item b)) is in two steps. In the first step, we have to show that, given two triangulated (resp., exact) morphisms of triangulated (resp., exact) derivators F, $F' : \mathbb{S} \to \mathbb{T}$ (resp., F, $F' : \underline{\mathbb{A}} \to \mathbb{T}$) and a triangulated (resp., exact) morphism $\mu : F \to F'$, according to def. 2.12, then the additive restriction $\mu| : F| \to F'|$ has the *property* that $\mu|_{\Delta_1} : F|_{\Delta_1} \to F'|_{\Delta_1}$ and $\mu|_\square : F|_\square \to F'|_\square$ are weakly triangulated (resp., weakly exact) functors. In the second step, vice–versa, we see that an additive morphism $\mu : F \to F'$ with such a property is indeed triangulated (resp., exact).

a) *1st step.* It is analogous to the first step in item b). We prefer to explicitely show this case because we will use it in the proof of Thm. 6.5 below.

2nd step. Suppose we are given an arbitrary additive morphism $F : \mathbb{S} \to \mathbb{T}$ such that F_{Δ_1} and $F_\square$ are weakly triangulated functors. We have to check that it preserves (homotopy) bicartesian objects according to the condition in the item b) of Definition 2.12. Since the derivators $\mathbb{S}$ and $\mathbb{T}$ have the *property* of being triangulated, thanks to the axiom **Der 7** it is sufficient to check that F preserves (homotopy) cocartesian objects.

Let us denote by ab the objects in the diagrams $\ulcorner$, $\lrcorner$ and $\square$ whose coordinates are (a, b). Accordingly, we denote $i_{ab,\ulcorner}$, $i_{ab,\lrcorner}$ and $i_{ab,\square}$ the functors defined by the object ab having coordinates (a, b), which send the diagram e into the diagrams $\ulcorner$, $\lrcorner$ and $\square$, respectively. We simply write i_{ab} when there is no possibility of confusion.

Remark that the pair of open / closed immersions $e \xhookrightarrow{i_{11}} \square \xhookleftarrow{i_\ulcorner} \ulcorner$ gives rise to a recollement of triangulated categories

$$\begin{tikzcd} \mathbb{S}(e) \arrow[r, "i_{11!}"] & \mathbb{S}(\square) \arrow[l, bend right, "{i_{11}}^?"'] \arrow[l, bend left, "{i_{11}}^*"] \arrow[r, "{i_\ulcorner}^*"] & \mathbb{S}(\ulcorner). \arrow[l, bend right, "i_{\ulcorner!}"'] \arrow[l, bend left, "i_{\ulcorner*}"] \end{tikzcd}$$

Thus, for any object X in $\mathbb{S}(\square)$, there is a distinguished triangle

$$i_{\ulcorner!}{i_\ulcorner}^*X \xrightarrow{\varepsilon} X \xrightarrow{\eta} i_{11!}{i_{11}}^?X \xrightarrow{\partial} \Sigma_\square i_{\ulcorner!}{i_\ulcorner}^*X.$$

Here, we respectively write ε, η, ∂ as for $\varepsilon^X_{i_\ulcorner}$, $\eta^X_{i_{11}}$, ∂^X. Remark that the connecting morphism ∂ is unique since $\mathsf{Hom}(\Sigma_\square i_{\ulcorner!}{i_\ulcorner}^*X, i_{11!}{i_{11}}^?X) = 0$ by adjunction and by the canonical isomorphism ${i_\ulcorner}^*i_{11!} = 0$, because of recollement.

Let us apply the *weakly* triangulated functor $F_\square$ and get a distinguished triangle in $\mathbb{T}(\square)$

$$F_\square i_{\ulcorner!}{i_\ulcorner}^*X \xrightarrow{F_\square\varepsilon} F_\square X \xrightarrow{F_\square\eta} F_\square i_{11!}{i_{11}}^?X \xrightarrow{w} \Sigma_\square F_\square i_{\ulcorner!}{i_\ulcorner}^*X.$$

Now, consider the canonical distinguished triangle associated to the object $F_\square X$,

$$i_{\ulcorner!}{i_\ulcorner}^*F_\square X \xrightarrow{\varepsilon^{F_\square}} F_\square X \xrightarrow{\eta^{F_\square}} i_{11!}{i_{11}}^?F_\square X \xrightarrow{\partial^{F_\square}} \Sigma_\square i_{\ulcorner!}{i_\ulcorner}^*F_\square X\,,$$

whose connecting morphism is also unique. Again, we use $\varepsilon^{F_\square}$, $\eta^{F_\square}$, $\partial^{F_\square}$ as for $\varepsilon^{F_\square X}_{i_\ulcorner}$, $\eta^{F_\square X}_{i_{11}}$, $\partial^{F_\square X}$.

Let us call $i_1 : e \to \Delta_1$ the open immersion which identifies e with 1 and $i_{\Delta_1} : \Delta_1 \to \square$ the open immersion which identifies 0 and 1 with 10 and 11, respectively. Having fixed this notation, we can write $i_{11} = i_{\Delta_1}i_1$. Let us fix an arbitrary object X lying in $\mathbb{S}(\Delta_1)$ or, respectively, in $\mathbb{S}(\square)$. By applying Lemma 4.4 twice to the diagram $I = e$ with $j = i_1$ and to the diagram $I = \Delta_1$ with $j = i_{\Delta_1}$, respectively, we get two canonical isomorphisms

$$\psi^X_{i_1i_1} : i_{1!}{i_1}^?F_{\Delta_1}X \xrightarrow{\sim} F_{\Delta_1}i_{1!}{i_1}^?X \qquad \text{and} \qquad \psi^X_{i_{\Delta_1}i_{\Delta_1}} : i_{\Delta_1!}{i_{\Delta_1}}^?F_\square X \xrightarrow{\sim} F_\square i_{\Delta_1!}{i_{\Delta_1}}^?X$$

such that the two commutation relations

$$\psi^X_{i_1i_1} \circ \eta^{F_{\Delta_1}X}_{i_1} = F_{\Delta_1}(\eta^X_{i_1}) \qquad \text{and} \qquad \psi^X_{i_{\Delta_1}i_{\Delta_1}} \circ \eta^{F_\square X}_{i_{\Delta_1}} = F_\square(\eta^X_{i_{\Delta_1}})$$

hold. Moreover, the functorial images of the two isomorphisms under ${i_1}^*$ and ${i_{\Delta_1}}^*$, respectively, furnish isomorphisms

$$\psi^X_{i_1} : {i_1}^?F_{\Delta_1}X \xrightarrow{\sim} F_e{i_1}^?X \qquad \text{and} \qquad \psi_{i_{\Delta_1}} : {i_{\Delta_1}}^?F_\square X \xrightarrow{\sim} F_{\Delta_1}{i_{\Delta_1}}^?X.$$

We can combine all these isomorphisms via the compositions of natural transformations

$$\begin{tikzcd} \mathbf{1} \arrow[r, "\eta_{\Delta_1}"] \arrow[rr, bend right, "\eta_{i_{11}}"'] & i_{\Delta_1!}{i_{\Delta_1}}^? \arrow[r, "i_{\Delta_1!}\eta_{i_1}{i_{\Delta_1}}^?"] & i_{\Delta_1!}i_{1!}{i_1}^?{i_{\Delta_1}}^? = i_{11!}{i_{11}}^? \end{tikzcd}$$

and

$$\begin{tikzcd} {i_{11}}^?i_{11!} = {i_1}^?{i_{\Delta_1}}^?i_{\Delta_1!}i_{1!} \arrow[r, "{i_1}^?\varepsilon_{i_{\Delta_1}}i_{1!}"] \arrow[rr, bend right, "\varepsilon_{i_{11}}"'] & {i_1}^?i_{1!} \arrow[r, "\varepsilon_{i_{\Delta_1}}"] & \mathbf{1} \end{tikzcd}$$

We get, canonically, an isomorphism $\psi_{i_1}^{i_{\Delta_1}^{?}X} \circ i_1{}^{?}(\psi_{i_{\Delta_1}}^X)$ as follows

$$\begin{aligned} i_{11}{}^{?}F_\Box X = (i_{\Delta_1}i_1)^{?}F_\Box X &= i_1{}^{?}i_{\Delta_1}{}^{?}F_\Box X \\ &\tilde{\to} i_1{}^{?}F_{\Delta_1}i_{\Delta_1}{}^{?}X \\ &\tilde{\to} F_e i_1{}^{?}i_{\Delta_1}{}^{?}X = F_e(i_{\Delta_1}i_1)^{?}X = F_e i_{11}{}^{?}X\,, \end{aligned}$$

which is canonically isomorphic to $\psi_{i_{11}}^X$.

Clearly, there is another functorial isomorphism $\psi_{i_{11}i_{11}}^X : i_{11!}i_{11}{}^{?}F_\Box X \tilde{\to} F_\Box i_{11!}i_{11}{}^{?}X$, explicitely given by the composition

$$i_{11!}i_{11}{}^{?}F_\Box X \xrightarrow[\sim]{i_{11!}(\psi_{i_{11}}^X)} i_{11!}F_e i_{11}{}^{?}X \xrightarrow[\sim]{\varphi^{i_{11},i_{11}{}^{?}X}} F_\Box i_{11!}i_{11}{}^{?}X.$$

Let us remark the important fact that the relation $\psi_{i_{11}i_{11}}^X \circ \eta_{i_{11}}^{F_\Box X} = F_\Box(\eta_{i_{11}}^X)$ holds. Indeed, we can form the following diagram

$$\begin{tikzcd}
F_\Box X \arrow[r, "\eta_{i_{\Delta_1}}^{F_\Box X}"] \arrow[ddd, equal] & i_{\Delta_1!}i_{\Delta_1}{}^{?}F_\Box X \arrow[rr, "i_{\Delta_1!}(\eta_{i_1}^{i_{\Delta_1}{}^{?}F_\Box X})"] \arrow[d, "\sim", "i_{\Delta_1!}(\psi_{i_{\Delta_1}}^X)"'] & & i_{\Delta_1!}i_{1!}i_1{}^{?}i_{\Delta_1}{}^{?}F_\Box X \arrow[r, equal] \arrow[d, "\sim"', "i_{\Delta_1!}i_{1!}i_1{}^{?}(\psi_{i_{\Delta_1}}^X)"] & i_{11!}i_{11}{}^{?}F_\Box X \arrow[ddd, "\sim"', "\psi_{i_{11}i_{11}}^X"] \\
& i_{\Delta_1!}F_{\Delta_1}i_{\Delta_1}{}^{?}X \arrow[rr, "i_{\Delta_1!}(\eta_{i_1}^{F_{\Delta_1}i_{\Delta_1}{}^{?}X})"] \arrow[d, equal] & & i_{\Delta_1!}i_{1!}i_1{}^{?}F_{\Delta_1}i_{\Delta_1}{}^{?}X \arrow[d, "\sim"', "i_{\Delta_1!}(\psi_{i_1i_1}^{i_{\Delta_1}^{?}X})"] & \\
& i_{\Delta_1!}F_{\Delta_1}i_{\Delta_1}{}^{?}X \arrow[rr, "i_{\Delta_1!}F_{\Delta_1}(\eta_{i_1}^{i_{\Delta_1}{}^{?}X})"] \arrow[d, "\sim", "\varphi^{i_{\Delta_1},i_{\Delta_1}{}^{?}X}"'] & & i_{\Delta_1!}F_{\Delta_1}i_{1!}i_1{}^{?}i_{\Delta_1}{}^{?}X \arrow[d, "\sim"', "\varphi^{i_{\Delta_1},i_{1!}i_1{}^{?}i_{\Delta_1}{}^{?}X}"] & \\
F_\Box X \arrow[r, "F_\Box\eta_{i_{\Delta_1}}^X"'] & F_\Box i_{\Delta_1!}i_{\Delta_1}{}^{?}X \arrow[rr, "F_\Box i_{\Delta_1!}(\eta_{i_1}^{i_{\Delta_1}{}^{?}X})"'] & & F_\Box i_{\Delta_1!}i_{1!}i_1{}^{?}i_{\Delta_1}{}^{?}X \arrow[r, equal] & F_\Box i_{11!}i_{11}{}^{?}X.
\end{tikzcd}$$

Let us consider the first square on the left. By definition, we have $\varphi^{i_{\Delta_1},i_{\Delta_1}{}^{?}X} \circ i_{\Delta_1!}(\psi_{i_{\Delta_1}}^X) = \psi_{i_{\Delta_1}i_{\Delta_1}}^X$ and we know (*cf.* lemma 4.4) that this morphism is an iso and makes the square commute.

The three squares in the center of the diagram also commute. Indeed, the square at the top commutes by functoriality of η_{i_1}, the square in the middle commutes by lemma 4.4 and the square at the bottom commutes because $\varphi^{i_{\Delta_1}}$ is a natural transformation.

The last square on the right also is commutative. This fact is easy to check by using the general relation proved in subsection 2.2 in our special case $v = i_1$ and $u = i_{\Delta_1}$, which gives the formula

$$\varphi^{i_{\Delta_1},\, i_{1!}X} \circ i_{\Delta_1!}(\varphi^{i_1,X}) = \varphi^{i_{11},\, X}\,,$$

for all X in $\mathbb{S}(e)$. By using this formula, we get a similar relation, valid for any X in $\mathbb{S}(e)$,

$$\begin{aligned}
\psi^X_{i_{11}} &= \varepsilon^{F_\square (i_{11})^? X}_{i_{11}} \circ (i_{11})^?[(\varphi^{i_{11},\,(i_{11})^? X})^{-1}] \circ (i_{11})^? F_e(\eta^X_{i_{11}}) \\
&= \varepsilon^{F_\square (i_{11})^? X}_{i_1} \circ {i_1}^?(\varepsilon^{i_{1!}F_\square (i_{11})^? X}_{i_{\Delta_1}}) \circ (i_{11})^? i_{\Delta_1 !}[(\varphi^{i_1,\,(i_{11})^? X})^{-1}] \circ \\
&\quad (i_{11})^?[(\varphi^{i_{\Delta_1},\, i_{1!}(i_{11})^? X})^{-1}] \circ (i_{11})^? F_e i_{\Delta_1 !}(\eta^{{i_{\Delta_1}}^? X}_{i_1}) \circ (i_{11})^? F_e(\eta^X_{i_{\Delta_1}}) \\
&= \varepsilon^{F_\square (i_{11})^? X}_{i_1} \circ {i_1}^?[(\varphi^{i_1,\,(i_{11})^? X})^{-1}] \circ {i_1}^?(\varepsilon^{F_{\Delta_1} i_{1!}{i_1}^? {i_{\Delta_1}}^? X}_{i_{\Delta_1}}) \circ \\
&\quad (i_{11})^? i_{\Delta_1 !} F_{\Delta_1}(\eta^{{i_{\Delta_1}}^? X}_{i_1}) \circ (i_{11})^?[(\varphi^{i_{\Delta_1},\, {i_{\Delta_1}}^? X})^{-1}] \circ (i_{11})^? F_e(\eta^X_{i_{\Delta_1}}) \\
&= \varepsilon^{F_\square (i_{11})^? X}_{i_1} \circ {i_1}^?[(\varphi^{i_1,\,(i_{11})^? X})^{-1}] \circ {i_1}^? F_{\Delta_1}(\eta^{{i_{\Delta_1}}^? X}_{i_1}) \circ \\
&\quad {i_1}^?(\varepsilon^{F_{\Delta_1} {i_{\Delta_1}}^? X}_{i_{\Delta_1}}) \circ (i_{11})^?[(\varphi^{i_{\Delta_1},\, {i_{\Delta_1}}^? X})^{-1}] \circ (i_{11})^? F_e(\eta^X_{i_{\Delta_1}}) \\
&= \psi^{{i_{\Delta_1}}^? X}_{i_1} \circ {i_1}^?(\psi^X_{i_{\Delta_1}}).
\end{aligned}$$

Altogheter, all these formulas show commutativity of the last square on the right.

Coming back to our fixed object X of $\mathbb{S}(\square)$, let us consider the morphism $\varphi^X_{i_\ulcorner i_\ulcorner}$: $i_{\ulcorner !}{i_\ulcorner}^* F_\square X \to F_\square i_{\ulcorner !}{i_\ulcorner}^* X$ that means, in our notations, the composition $\varphi^{i_\ulcorner,\, {i_\ulcorner}^* X} \circ i_{\ulcorner !}[(\varphi^X_{i_\ulcorner})^{-1}]$. We can see that the morphism $\varepsilon^{F_\square}$ factors through $\varphi^X_{i_\ulcorner i_\ulcorner}$. Indeed, we can compute

$$\begin{aligned}
F_\square(\varepsilon^X_{i_\ulcorner}) \circ \varphi^X_{i_\ulcorner i_\ulcorner} &= F_\square(\varepsilon^X_{i_\ulcorner}) \circ \varphi^{i_{\ulcorner !},\, {i_\ulcorner}^* X} \circ i_{\ulcorner !}[(\varphi^X_{i_\ulcorner})^{-1}] \\
&= F_\square(\varepsilon^X_{i_\ulcorner}) \circ \varepsilon^{F_\square i_{\ulcorner !}{i_\ulcorner}^* X}_{i_\ulcorner} \circ i_{\ulcorner !}(\varphi^{i_{\ulcorner !}{i_\ulcorner}^* X}_{i_\ulcorner}) \circ i_{\ulcorner !} F_\ulcorner(\eta^{{i_\ulcorner}^* X}_{i_\ulcorner}) \circ i_{\ulcorner !}[(\varphi^X_{i_\ulcorner})^{-1}] \\
&= \varepsilon^{F_\square X}_{i_\ulcorner} \circ i_{\ulcorner !}{i_\ulcorner}^* F_\square(\varepsilon^X_{i_\ulcorner}) \circ i_{\ulcorner !}(\varphi^{i_{\ulcorner !}{i_\ulcorner}^* X}_{i_\ulcorner}) \circ i_{\ulcorner !} F_\ulcorner(\eta^{{i_\ulcorner}^* X}_{i_\ulcorner}) \circ i_{\ulcorner !}[(\varphi^X_{i_\ulcorner})^{-1}] \\
&= \varepsilon^{F_\square X}_{i_\ulcorner} \circ i_{\ulcorner !}(\varphi^X_{i_\ulcorner}) \circ i_{\ulcorner !} F_\ulcorner {i_\ulcorner}^*(\varepsilon^X_{i_\ulcorner}) \circ i_{\ulcorner !} F_\ulcorner(\eta^{{i_\ulcorner}^* X}_{i_\ulcorner}) \circ i_{\ulcorner !}[(\varphi^X_{i_\ulcorner})^{-1}] \\
&= \varepsilon^{F_\square X}_{i_\ulcorner}.
\end{aligned}$$

Let us consider an extension of this factorization to a morphism of distinguished triangles of $\mathbb{T}(\square)$

$$\begin{tikzcd}
i_{\ulcorner !}{i_\ulcorner}^* F_\square X \arrow[r, "\varepsilon^{F_\square}"] \arrow[d, "\varphi^X_{i_\ulcorner i_\ulcorner}"'] & F_\square X \arrow[r, "\eta^{F_\square}"] \arrow[d, equal, "\mathbf{1}"] & i_{11!}{i_{11}}^? F_\square X \arrow[r, "\partial^{F_\square}"] \arrow[d, dashed, "\psi^X_\square"] & \Sigma_\square i_{\ulcorner !}{i_\ulcorner}^* F_\square X \arrow[d, "\Sigma\varphi^X_{i_\ulcorner i_\ulcorner}"] \\
F_\square i_{\ulcorner !}{i_\ulcorner}^* X \arrow[r, "F_\square \varepsilon"] & F_\square X \arrow[r, "F_\square \eta"] & F_\square i_{11!}{i_{11}}^? X \arrow[r, "w"] & \Sigma_\square F_\square i_{\ulcorner !}{i_\ulcorner}^* X.
\end{tikzcd}$$

The extension morphism $\psi^X_\square$ is *unique* making commute the central square because the group

$$\begin{aligned}
\mathsf{Hom}(\Sigma_\square i_{\ulcorner !}{i_\ulcorner}^* F_\square X, F_\square i_{11!}{i_{11}}^? X) &= \mathsf{Hom}(\Sigma_\square i_{\ulcorner !}{i_\ulcorner}^* F_\square X, i_{11!}{i_{11}}^? F_\square X) \\
&= \mathsf{Hom}(i_{\ulcorner !}\Sigma_\square {i_\ulcorner}^* F_\square X, i_{11!}{i_{11}}^? F_\square X) \\
&= \mathsf{Hom}(\Sigma_\square {i_\ulcorner}^* F_\square X, {i_\ulcorner}^* i_{11!}{i_{11}}^? F_\square X) \\
&= 0
\end{aligned}$$

is canonically trivial via the canonical isomorphism $\psi^X_{i_{11} i_{11}}$, adjunction and the canonical isomorphism ${i_\ulcorner}^* i_{11!} = 0$, thanks to the recollement axioms. Hence, we must canonically identify $\psi^X_\square$ with the isomorphism $\psi^X_{i_{11} i_{11}}$. It follows that $\varphi^X_{i_\ulcorner i_\ulcorner}$ is invertible, too. Therefore, $\varepsilon^X_{i_\ulcorner}$ is invertible if and only if $\varepsilon^{F_\square X}_{i_\ulcorner}$ is.

It is clear that a completely analogous proof (but with tridimensional diagrams!) works to prove that the canonical morphism $\varepsilon_{i_\ulcorner}^{\mu_\square^X} : i_{\ulcorner !} i_\ulcorner{}^* \mu_\square^X \to \mu_\square^X$ is invertible anytime $\mu : F \to F'$ is an *additive* morphism of triangulated morphisms of triangulated derivators F , F' such that μ_{Δ_1} and $\mu_\square$ are weakly triangulated functors.

b) *1st step.* Let us show the claim for all $I \in \mathscr{D}ia$. We begin with an exact morphism of derivators $F : \underline{\mathbb{A}} \to \mathbb{S}$ as in the hypothesis. We want to show that the functor $F|_I = F_I : \underline{\mathbb{A}}(I) \to \mathbb{S}(I)$ is weakly exact, for all diagrams I.

Indeed, for an arbitrary I, given a conflation ε in $\underline{\mathbb{A}}(I) = \underline{\mathbb{A}}^I(e)$

$$X \rightarrowtail Y \twoheadrightarrow Z \,,$$

there is a bicartesian square

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d, two heads] & Y \arrow[d, two heads] \\ 0 \arrow[r, tail] & Z \end{tikzcd}$$

lying in $\underline{\mathsf{Hom}}(\square^\circ, \underline{\mathbb{A}}^I(e))$. Since we can write this category as $\underline{\mathsf{Hom}}(\Delta_1^\circ, \underline{\mathbb{A}}^I(\Delta_1))$, thanks to the axiom **Der 5** there exists a bicartesian object S in $\mathbb{A}^I(\square)$ whose diagram is the square above.

Clearly, we have a canonical iso $\varepsilon^S : i_{\ulcorner !} i_\ulcorner{}^* S \xrightarrow{\sim} S$. For the sake of simplicity in the notation, here we write $i_{i_\ulcorner}$, $i_\square$, etc., instead of the correct ones $i^I_{i_\ulcorner} = i_{I \times i_\ulcorner}$, $i^I_\square = i_{I\times\square}$, etc. Since the morphism F is supposed to be exact, we get an object $F^I_\square S$ in $\mathbb{S}^I(\square)$ having a commutative square as diagram

$$\mathsf{dia}^I_\square(F^I_\square S) = F^I_e(\mathsf{dia}^I_\square S) \quad = \quad \begin{tikzcd} F^I_e X \arrow[r] \arrow[d] & F^I_e Y \arrow[d] \\ 0 \arrow[r] & F^I_e Z \end{tikzcd} \,,$$

such that $\varepsilon^{F^I_\square S} : i_{\ulcorner !} i_\ulcorner{}^* F^I_\square S \to F^I_\square S$ is invertible. Our local claim is that there exists a distinguished triangle in $\mathbb{S}^I(e)$

$$F^I_e X \longrightarrow F^I_e Y \longrightarrow F^I_e Z \xrightarrow{\ w\ } \Sigma F^I_e X \,,$$

for some morphism $w : F^I_e Z \to \Sigma F^I_e X$, *i.e.*, that the functor F^I_e is weakly exact.

Let us fix some notation. We write $\square\!\square$ for the diagram $\Delta_2 \times \Delta_1$ and $\sqsubset\!\urcorner$ for the diagram $\square\!\square$ after we erase the object $(1,1)$. There are obvious fully faithful inclusions $i_\square : \square \to \sqsubset\!\urcorner$, $i_\ulcorner : \ulcorner \to \square$ and $i_{\sqsubset\!\urcorner} : \sqsubset\!\urcorner \to \square\!\square$, injective on objects. Moreover, we define other inclusions $l_\square : \square \to \square\!\square$ and $r_\square : \square \to \square\!\square$ defined by the evident overlap of the image of the square over the small squares on the left and on the right of the two-square, respectively. These functors further induce other inclusions $l_\ulcorner : \ulcorner \to \square\!\square$ and $r_\ulcorner : \ulcorner \to \square\!\square$, which are the compositions $l_\square i_\ulcorner$ and $r_\square i_\ulcorner$, respectively. We also consider the 'global' inclusion $g_\square : \square \to \square\!\square$ mapping the square on the exterior bord of the two-square and the inclusion $i_{\ulcorner\,,\,\sqsubset\!\urcorner} : \ulcorner \to \sqsubset\!\urcorner$ mapping the diagram $\ulcorner$ to the right corner of the diagram $\sqsubset\!\urcorner$.

Let us define the object $P := i_{\sqsubset\!\urcorner !} i_{\square *} F^I_\square S$. This is a *polycartesian* object of $\mathbb{S}^I(\square\!\square)$ in the sense of Maltsiniotis [13], *i.e.*, there are isomorphisms

$$\varepsilon^{l_\square{}^* P} : i_{\ulcorner !} i_\ulcorner{}^* l_\square{}^* P \xrightarrow{\sim} l_\square{}^* P, \qquad \varepsilon^{r_\square{}^* P} : i_{\ulcorner !} i_\ulcorner{}^* r_\square{}^* P \xrightarrow{\sim} r_\square{}^* P, \qquad \varepsilon^{g_\square{}^* P} : i_{\ulcorner !} i_\ulcorner{}^* g_\square{}^* P \xrightarrow{\sim} g_\square{}^* P.$$

It suffices to check the first and the second isos. We compute

$$
\begin{array}{rcl}
i_{\ulcorner !}{i_{\ulcorner}}^{*}{l_{\square}}^{*}P & = & i_{\ulcorner !}{i_{\ulcorner}}^{*}{i_{\square}}^{*}{i_{\square\!\urcorner}}^{*}i_{\square\!\urcorner !}i_{\square *}F^{I}_{\square}S \\
& = & i_{\ulcorner !}{i_{\ulcorner}}^{*}{i_{\square}}^{*}i_{\square *}F^{I}_{\square}S \\
& = & i_{\ulcorner !}{i_{\ulcorner}}^{*}F^{I}_{\square}S \\
& \tilde{\to} & F^{I}_{\square}S \\
& = & {i_{\square}}^{*}i_{\square *}F^{I}_{\square}S \\
& = & {i_{\square}}^{*}{i_{\square\!\urcorner}}^{*}i_{\square\!\urcorner !}i_{\square *}F^{I}_{\square}S \\
& = & {l_{\square}}^{*}P.
\end{array}
$$

Here we use the relation $l_{\square} = i_{\square\!\urcorner}i_{\square}$ in the first and the last equalities. The crucial iso $\varepsilon^{F^{I}_{\square}S}$ is the fourth, which is our hypothesis. As for the square on the right, let us compute

$$
\begin{array}{rcl}
i_{\ulcorner !}{i_{\ulcorner}}^{*}{r_{\square}}^{*}P & = & i_{\ulcorner !}{i_{\ulcorner}}^{*}{r_{\square}}^{*}i_{\square\!\urcorner !}i_{\square *}F^{I}_{\square}S \\
& = & i_{\ulcorner !}{r_{\ulcorner}}^{*}i_{\square\!\urcorner !}i_{\square *}F^{I}_{\square}S \\
& = & i_{\ulcorner !}{i_{\ulcorner\,,\,\square\!\urcorner}}^{*}{i_{\square\!\urcorner}}^{*}i_{\square\!\urcorner !}i_{\square *}F^{I}_{\square}S \\
& = & i_{\ulcorner !}{i_{\ulcorner\,,\,\square\!\urcorner}}^{*}i_{\square *}F^{I}_{\square}S \\
& \tilde{\to} & {r_{\square}}^{*}i_{\square\!\urcorner !}i_{\square *}F^{I}_{\square}S \\
& = & {r_{\square}}^{*}P.
\end{array}
$$

Here, the only non trivial iso comes from the invertible natural transformation $i_{\ulcorner !}{i_{\ulcorner\,,\,\square\!\urcorner}}^{*} \tilde{\to} {r_{\square}}^{*}i_{\square\!\urcorner !}$. To see why this transformation is invertible, let us consider the co-cartesian square in the category $\mathscr{D}ia$

$$
\begin{tikzcd}
\ulcorner \arrow[r, "i_{\ulcorner}"] \arrow[d, "i_{\ulcorner\,,\,\square\!\urcorner}"'] & \square \arrow[d, "r_{\square}"] \\
\square\!\urcorner \arrow[r, "i_{\square\!\urcorner}"'] & \square\square.
\end{tikzcd}
$$

By the "commentaires" after the axiom **Der 7** in [13] or, in greater detail, by the dual of Prop. 6.9 in [3, Prop. 6.9] the required isomorphism follows. The local claim follows by the description of the triangulated structure over $\mathbb{S}^{I}(e)$ as, *e.g.* , in [13].

Let us consider now an exact morphism $\mu : F \to F'$, where $F' : \underline{\mathbb{A}} \to \mathbb{S}$ is another exact morphism of derivators. We desire to show that the functor $\mu_{I} : F_{I} \to F'_{I}$ actually is a weakly exact morphism. Indeed, given a conflation ε in $\underline{\mathbb{A}}(I) = \underline{\mathbb{A}}^{I}(e)$

$$X \rightarrowtail Y \twoheadrightarrow Z\,,$$

thanks to the axiom **Der 5** there is a bicartesian object S in $\mathbb{A}^{I}(\square)$ whose diagram is as above. Again, we have a canonical iso $\varepsilon^{S} : i_{\ulcorner !}{i_{\ulcorner}}^{*}S \tilde{\to} S$. We already know that there are isomorphisms $\varepsilon^{F^{I}_{\square}S} : i_{\ulcorner !}{i_{\ulcorner}}^{*}F^{I}_{\square}S \tilde{\to} F^{I}_{\square}S$ and $\varepsilon^{F'^{I}_{\square}S} : i_{\ulcorner !}{i_{\ulcorner}}^{*}F'^{I}_{\square}S \tilde{\to} F'^{I}_{\square}S$ because F and F' are exact.

Since the morphism of derivators $\mu : F \to F'$ is supposed to be exact, it induces a morphism ${\mu^I_\square}^S : F^I_\square S \to F'^I_\square S$ in $\mathbb{S}^I(\square)$, whose diagram in $\mathbb{S}^I(e)$ is

$$\mathsf{dia}^I_\square({\mu^I_\square}^S) = {\mu^I_e}^{\mathsf{dia}^I_\square S} \quad = \quad \begin{tikzcd} F^I_e X \arrow[rr] \arrow[dr] \arrow[dd] & & F^I_e Y \arrow[dr] \arrow[dd] & \\ & 0 \arrow[rr] \arrow[dd] & & F^I_e Z \arrow[dd] \\ F'^I_e X \arrow[rr, dashed] \arrow[dr] & & F'^I_e Y \arrow[dr] & \\ & 0 \arrow[rr] & & F'^I_e Z \end{tikzcd} \quad ,$$

such that the morphism $i_{\ulcorner !} {i_{\ulcorner}}^* {\mu^I_\square}^S \xrightarrow{\sim} {\mu^I_\square}^S$ is invertible (*cf.* item c) of Def. 2.12).

Our local claim is that there exists a morphism of distinguished triangles in $\mathbb{S}^I(e)$

$$\begin{tikzcd} F^I_e X \arrow[r] \arrow[d, "{\mu^I_e}^X"'] & F^I_e Y \arrow[r] \arrow[d, "{\mu^I_e}^Y"'] & F^I_e Z \arrow[r, "w"] \arrow[d, "{\mu^I_e}^Z"] & \Sigma F^I_e X \arrow[d, "\Sigma {\mu^I_e}^X"] \\ F'^I_e X \arrow[r] & F'^I_e Y \arrow[r] & F'^I_e Z \arrow[r, "w'"] & \Sigma F'^I_e X \,, \end{tikzcd}$$

for some morphisms $w : F^I_e Z \to \Sigma F^I_e X$, $w' : F'^I_e Z \to \Sigma F'^I_e X$ and an isomorphism $\Sigma {\mu^I_e}^X \circ w \xrightarrow{\sim} w' \circ {\mu^I_e}^Z$, *i.e.*, that the morphism of functors μ^I_e is exact.

As above, let us define the polycartesian objects $P := i_{\square\!\!\lrcorner !} i_{\square *} F^I_\square S$ and $P' := i_{\square\!\!\lrcorner !} i_{\square *} F'^I_\square S$. We also define a morphism $\mu^P : P \to P'$ to be $i_{\square\!\!\lrcorner !} i_{\square *} {\mu^I_\square}^S$. This morphism is *polycartesian* in the following sense: there are isomorphisms

$$\varepsilon^{{l_\square}^* \mu^P} : i_{\ulcorner !} {i_\ulcorner}^* {l_\square}^* \mu^P \xrightarrow{\sim} {l_\square}^* \mu^P,\ \varepsilon^{{r_\square}^* \mu^P} : i_{\ulcorner !} {i_\ulcorner}^* {r_\square}^* \mu^P \xrightarrow{\sim} {r_\square}^* \mu^P,\ \varepsilon^{{g_\square}^* \mu^P} : i_{\ulcorner !} {i_\ulcorner}^* {g_\square}^* \mu^P \xrightarrow{\sim} {g_\square}^* \mu^P.$$

It suffices to check the first and the second isos. We compute

$$\begin{aligned} i_{\ulcorner !} {i_\ulcorner}^* {l_\square}^* \mu^P &= i_{\ulcorner !} {i_\ulcorner}^* {i_\square}^* {i_{\square\!\!\lrcorner}}^* i_{\square\!\!\lrcorner !} i_{\square *} {\mu^I_\square}^S \\ &= i_{\ulcorner !} {i_\ulcorner}^* {i_\square}^* i_{\square *} {\mu^I_\square}^S \\ &= i_{\ulcorner !} {i_\ulcorner}^* {\mu^I_\square}^S \\ &\xrightarrow{\sim} {\mu^I_\square}^S \\ &= {i_\square}^* i_{\square *} {\mu^I_\square}^S \\ &= {i_\square}^* {i_{\square\!\!\lrcorner}}^* i_{\square\!\!\lrcorner !} i_{\square *} {\mu^I_\square}^S \\ &= {l_\square}^* \mu^P. \end{aligned}$$

Here we use the relation $l_\square = i_{\square\!\!\lrcorner} i_\square$ in the first and the last equalities. The crucial iso $i_{\ulcorner !} {i_\ulcorner}^* {\mu^I_\square}^S \xrightarrow{\sim} {\mu^I_\square}^S$ is the fourth, which is our hypothesis. As for the square on the right, let

us compute

$$\begin{aligned}
i_{\ulcorner !} {i_{\ulcorner}}^* {r_{\square}}^* \mu^P &= i_{\ulcorner !} {i_{\ulcorner}}^* {r_{\square}}^* i_{\square\!\!\ulcorner !} i_{\square *} {\mu_{\square}^{I}}^{S} \\
&= i_{\ulcorner !} {r_{\ulcorner}}^* i_{\square\!\!\ulcorner !} i_{\square *} {\mu_{\square}^{I}}^{S} \\
&= i_{\ulcorner !} {i_{\ulcorner, \square\!\!\ulcorner}}^* {i_{\square\!\!\ulcorner}}^* i_{\square\!\!\ulcorner !} i_{\square *} {\mu_{\square}^{I}}^{S} \\
&= i_{\ulcorner !} {i_{\ulcorner, \square\!\!\ulcorner}}^* i_{\square *} {\mu_{\square}^{I}}^{S} \\
&\tilde{\to} {r_{\square}}^* i_{\square\!\!\ulcorner !} i_{\square *} {\mu_{\square}^{I}}^{S} \\
&= {r_{\square}}^* \mu^P .
\end{aligned}$$

Here, the isos we are using are in complete analogy with those at the corresponding point in the proof of the object case. The local claim follows by the description of the triangulated structure over $\mathbb{S}^I(e)$ as, *e.g.* , in [13].

2nd step. Since Axiom **Der 7** holds for the derivator $\mathbb{S}$, it suffices to show that, given any cocartesian (hence cartesian) square $X \in \underline{\mathbb{A}}(\square)$ as in Definition 2.12, part a), whose diagram is

$$\mathsf{dia}_{\square} X \quad = \quad \begin{tikzcd} i_{00}^* X \arrow[r, tail] \arrow[d, two heads] & i_{01}^* X \arrow[d, two heads] \\ i_{10}^* X \arrow[r, tail] & i_{11}^* X \end{tikzcd} \quad ,$$

the adjunction morphism

$$\varepsilon^{F_{\square} X} : i_{\ulcorner !} {i_{\ulcorner}}^* F_{\square} X \to F_{\square} X$$

is invertible.

Let us start with a slightly more general object X in $\underline{\mathbb{A}}(\square)$, whose diagram is a commutative square with inflations as horizontal arrows and deflations as vertical arrows, which is not required to be cartesian, nor cocartesian. Let us suppose that the adjoint morphism $\varepsilon^X : i_{\ulcorner !} {i_{\ulcorner}}^* X \to X$ is a deflation. This is not restrictive. Indeed, it occurs under the mild additional hypothesis that the exact categories we are considering have split idempotents, or, which is the case we will be dealing with, when the square object X is bicartesian. In fact, the object ${i_{11}}^* i_{\ulcorner !} {i_{\ulcorner}}^* X$ is the homotopy colimit $\mathsf{hocolim}_{\ulcorner} {i_{\ulcorner}}^* X$.

Let us explain this, in general. Let us consider the category $11 \backslash \ulcorner$, defined by the inclusion $i_{\ulcorner} : \ulcorner \to \square$. There is a (noncommutative) square

$$\begin{tikzcd} 11\backslash\ulcorner \arrow[r, "\simeq", "j"'] \arrow[d, "p_{11\backslash\ulcorner}"'] & \ulcorner \arrow[d, "i_{\ulcorner}"] \arrow[dl, "p_{\ulcorner}"] \\ e \arrow[r, "i_{11,\square}"'] & \square \end{tikzcd} \quad ,$$

where the forgetful functor j is an isomorphism of categories. By Axiom **Der 4**, we find a functorial isomorphism of functors

$$c'_{11\backslash\ulcorner} : (p_{\ulcorner})_! = (p_{11\backslash\ulcorner})_! j^* \xrightarrow{\sim} i_{11,\square}^* i_{\ulcorner !} \ .$$

Moreover, it is not difficult to see that there is a *canonical* iso

$$i_{01,\ulcorner}^* Z \coprod_{i_{00,\ulcorner}^* Z} i_{10,\ulcorner}^* Z \tilde{\to} p_{\ulcorner !} Z \ ,$$

for any object Z in $\underline{\mathbb{A}}(\ulcorner)$. Thus, by the uniqueness of push-outs in exact categories it follows that the (unique) universal morphism ${i_{11}}^*(\varepsilon^X)$ is a deflation, if the exact category $\underline{\mathbb{A}}(e) = \mathcal{A}$ has splitting idempotents, or even an iso, if the the square is supposed to be cocartesian.

Let us consider the conflation induced by the deflation ε^X,

$$\Omega i_{11!}{i_{11}}^? \rightarrowtail i_{\ulcorner !}{i_{\ulcorner}}^* X \twoheadrightarrow X \;,$$

and apply the *weak* ∂-functor $F_\square$. After shifting, we get a distinguished triangle

$$F_\square i_{\ulcorner !}{i_{\ulcorner}}^* X \longrightarrow F_\square X \longrightarrow F_\square i_{11!}{i_{11}}^? X \longrightarrow \Sigma_\square F_\square i_{\ulcorner !}{i_{\ulcorner}}^* X \;,$$

The proof goes on along the same lines of the proof of item a), with the difference that this time we will apply lemma 4.5 anytime we were applying lemma 4.4. □

5. The Universal Property

In this section we give a proof of the main theorem 2.13. Let us put together some trivial observations at first.

Lemma 5.1. *Let $\mathbb{E}$ be an exact (resp., triangulated) derivator. Fix an arbitrary diagram I in $\mathscr{D}ia_{\mathsf{f}}$. Then, the sequence of exact (resp., triangulated) categories defined by*

$$\mathsf{E}_n^I := \mathbb{E}(C_n \times I) \;,\; n \in \mathbb{N} \;,$$

with the induced functors defined by

$$u_I^* := (u \times \mathbf{1}_I)^* : \mathsf{E}_n^I \to \mathsf{E}_m^I \;,$$

for any 1-morphism $u : C_m \to C_n$ in $\mathscr{C}ubes$, and the natural transformations defined by

$$\alpha_I^* := (\alpha \times \mathbf{1}_I)^* : v_I^* \Rightarrow u_I^* \;,$$

for any 2-morphism $\alpha : u \Rightarrow v$ in $\mathscr{C}ubes$, give rise to an exact (resp., triangulated) tower E^I such that $\mathsf{E}^I(C_n) = \mathsf{E}_n^I$.

Proof. It reduces to a straightforward checking that the axioms of an exact (resp. triangulated) tower hold for E^I. □

Lemma 5.2. *Let $F : \mathbb{D} \to \mathbb{E}$ be an exact (resp., triangulated) morphism of exact (resp., triangulated) derivators. Fix an arbitrary diagram I in $\mathscr{D}ia_{\mathsf{f}}$. Then, the sequence of associated morphisms defined by*

$$F_n^I := F_{C_n \times I} : \mathsf{D}_n^I \to \mathsf{E}_n^I \;,\; n \in \mathbb{N} \;,$$

with the induced isomorphisms of functors defined by

$$\varphi_u^I := \varphi_{u \times \mathbf{1}_I} : F_m^I u_I^* \tilde{\to} u_I^* F_n^I \;,$$

for any 1-morphism $u : C_m \to C_n$ in $\mathscr{C}ubes$, gives rise to a morphism of exact (resp., triangulated) towers $F^I : \mathsf{D}^I \to \mathsf{E}^I$ such that $F_n^I = F_{C_n \times I}$.

Proof. Again, it is straightforward to check that the axioms of an exact (resp. triangulated) morphism of towers hold for F^I. □

Let I be any diagram of $\mathscr{D}ia_{\mathsf{f}}$. Let us associate an additive (resp., exact) tower $\underline{\mathsf{A}^I}$ with any additive (resp., exact) category $\mathcal{A}$, according to the following definition

$$\underline{\mathsf{A}^I}(C_n) := \underline{\mathsf{Hom}}(C_n^\circ, \underline{\mathsf{Hom}}(I^\circ, \mathcal{A})) \;,\; n \in \mathbb{N} \;.$$

Lemma 5.3. *Let $\mathcal{A}$ be an exact category and $\mathbb{E}$ an exact derivator. Suppose that $F : \underline{\mathbb{A}} \to \mathbb{E}$ is an exact morphism of exact derivators. Fix an arbitrary diagram I in $\mathscr{D}ia_{\mathsf{f}}$. Then, there is an induced exact morphism of exact towers*

$$F^I : \underline{\mathsf{A}^I} \to \mathsf{E}^I$$

where $\underline{\mathsf{A}^I}$ equals $\underline{\mathsf{A}}^I$ (according to the notations of Lemma 5.1).

Proof. For each $n \in \mathbb{N}$, we have

$$\begin{array}{rcl} \underline{\mathsf{A}}^I(C_n) & = & \underline{\mathbb{A}}(C_n \times I) \\ & = & \underline{\mathsf{Hom}}((C_n \times I)^\circ, \mathcal{A}) \\ & = & \underline{\mathsf{Hom}}(C_n^\circ \times I^\circ, \mathcal{A}) \\ & = & \underline{\mathsf{Hom}}(C_n^\circ, \underline{\mathsf{Hom}}(I^\circ, \mathcal{A})) \\ & = & \underline{\mathsf{A}^I}(C_n) \, . \end{array}$$

Now use Lemma 5.1 and Lemma 5.2. □

We can now prove the main Theorem 2.13.

Proof. Let us consider an exact morphism of derivators $F : \underline{\mathbb{A}} \to \mathbb{E}$. We functorially construct an object $\widetilde{F}$ lying in $\underline{\mathcal{H}om}_{tr}(\mathbb{D}_{\mathcal{A}}, \mathbb{E})$ such that $can^*(\widetilde{F}) = \widetilde{F} \circ can$ is F (up to a unique iso). By Lemma 5.3, if we fix a diagram I we have the induced exact morphism of towers $F^I : \underline{\mathsf{A}^I} \to \mathsf{E}^I$. This is an object in the category $\underline{\mathcal{H}om}_{ex}(\underline{\mathsf{A}^I}\,, \mathsf{E}^I)$. By Keller's theorem 3.3, this morphism extends uniquely to a morphism of triangulated towers $\widetilde{F^I} : \mathsf{D}_{\mathcal{A}^I} \to \mathsf{E}^I$. Here $\mathsf{D}_{\mathcal{A}^I}$ is a tower that associates the bounded derived category $\mathcal{D}^b(\underline{\mathsf{Hom}}(C_n^\circ, \underline{\mathsf{Hom}}(I^\circ, \mathcal{A}))$ with any cube C_n and equals the tower $\mathsf{D}^I_{\mathcal{A}}$ which is defined by the relation $\mathsf{D}^I_{\mathcal{A}}(C_n) = \mathcal{D}^b(\underline{\mathsf{Hom}}((C_n \times I)^\circ, \mathcal{A}))$. Thus, the morphism $\widetilde{F^I} : \mathsf{D}_{\mathcal{A}^I} \to \mathsf{E}^I$ identifies with a morphism $\widetilde{F}^I : \mathsf{D}^I_{\mathcal{A}} \to \mathsf{E}^I$. The *base* of the morphism $\widetilde{F}^I$ (*i.e.*, the evaluation at C_0) gives us a triangulated functor $(\widetilde{F}^I)_0 : \mathbb{D}_{\mathcal{A}}(I) \to \mathbb{E}(I)$.

It remains to check that, by letting I running in $\mathscr{D}ia_{\mathsf{f}}$, we get a triangulated morphism of derivators $\widetilde{F} : \mathbb{D}_{\mathcal{A}} \to \mathbb{E}$, defined by the equality $\widetilde{F}_I := (\widetilde{F}^I)_0$, whose image in $\underline{\mathcal{H}om}_{ex}(\underline{\mathbb{A}}, \mathbb{E})$ is F. Notice that, by the construction, the composition $\widetilde{F}_I \circ can_I$ is F_I, for all diagrams I (up to a canonical iso). So, we can start with the second item in Definition 2.7. Let $u : I \to J$ be a 1-morphism of finite diagrams. Since F is a morphism of derivators, there is a square

$$\begin{tikzcd} \underline{\mathbb{A}}(J) \arrow[r, "u^*"] \arrow[d, "F_J"'] & \underline{\mathbb{A}}(I) \arrow[d, "F_I"] \\ \mathbb{E}(J) \arrow[r, "u^*"'] & \mathbb{E}(I) \end{tikzcd} \quad ,$$

which is commutative (up to a unique iso) by the definition, *i.e.*, there is an isomorphism of functors $\varphi_u : F_I u^* \tilde{\to} u^* F_J$. Now, the previous square is (canonically isomorphic to) the base of the following commutative square of towers

$$\begin{tikzcd} \underline{\mathsf{A}^J} \arrow[r, "u^{*\wedge}"] \arrow[d, "F^J"'] & \underline{\mathsf{A}^I} \arrow[d, "F^I"] \\ \mathsf{E}^J \arrow[r, "u^{*\wedge}"'] & \mathsf{E}^I \end{tikzcd} \quad ,$$

where $u^{*\wedge}$ is the induced exact morphism of towers $\underline{\mathsf{Hom}}((-)^\circ, u^*)$. By Theorem 3.3 the morphisms F^J and F^I uniquely extend to morphisms $\widetilde{F^J}$ and $\widetilde{F^I}$. Since the morphism $u^{*\wedge}$ is exact it extend to the derived morphism $\mathbf{R}u^{*\wedge}$. So, we get a square of triangulated towers

$$\begin{tikzcd} \mathsf{D}_{\mathcal{A}^J} \arrow[r, "\mathbf{R}u^{*\wedge}"] \arrow[d, "\widetilde{F^J}"'] & \mathsf{D}_{\mathcal{A}^I} \arrow[d, "\widetilde{F^I}"] \\ \mathsf{E}^J \arrow[r, "u^{*\wedge}"'] & \mathsf{E}^I \end{tikzcd} ,$$

which is equal to the following commutative square

$$\begin{tikzcd} \mathsf{D}_{\mathcal{A}}^J \arrow[r, "\mathbf{R}u^{*\wedge}"] \arrow[d, "\widetilde{F}^J"'] & \mathsf{D}_{\mathcal{A}}^I \arrow[d, "\widetilde{F}^I"] \\ \mathsf{E}^J \arrow[r, "u^{*\wedge}"'] & \mathsf{E}^I \end{tikzcd} ,$$

whose base (evaluation at C_0) is a square of triangulated categories

$$\begin{tikzcd} \mathbb{D}_{\mathcal{A}}(J) \arrow[r, "u^*:=\mathbf{R}u^*"] \arrow[d, "\widetilde{F}_J"'] & \mathbb{D}_{\mathcal{A}}(I) \arrow[d, "\widetilde{F}_I"] \\ \mathbb{E}(J) \arrow[r, "u^*"'] & \mathbb{E}(I) \end{tikzcd} .$$

The functor $\mathbf{R}u^*$ exists since $u^* : \underline{\mathbb{A}}(J) \to \underline{\mathbb{A}}(I)$ is an exact functor of exact categories.

It remains to check that this square is commutative up to a unique iso. Let $\widetilde{\varphi}_u : \widetilde{F}_I \mathbf{R}u^* \to u^* \widetilde{F}_J$ be the morphism of functors induced by φ_u via the fully faithful functor can_J. We apply item b) in [10, Lemme 2], where we identify the functor G with $\widetilde{F}_I \mathbf{R}u^*$, the functor G' with $u^* \widetilde{F}_J$ and the class of objects $\mathcal{X}$ with the set of objects of the category $\underline{\mathbb{A}}(J)$. Indeed, the image of the subcategory $\underline{\mathbb{A}}(J)$ generates $\mathbb{D}_{\mathcal{A}}(J)$ as a triangulated category. Since the restriction of the functor $\widetilde{\varphi}_u$ to the subcategory $\underline{\mathbb{A}}(J)$ equals the isomorphism φ_u, it follows by the cited Lemma that $\widetilde{\varphi}_u : \widetilde{F}_I \mathbf{R}u^* \to u^* \widetilde{F}_J$ is an isomorphism of functors. It is then clear how to check the remaining axioms of a morphism of derivators.

The additive morphism $\widetilde{F}$ extends to a triangulated morphism. Indeed, the functor $\widetilde{F}_I$ is triangulated with respect to the canonical triangulated structures of the categories $\mathbb{D}_{\mathcal{A}}(I)$ and $\mathbb{E}(I)$, for all diagrams I. Therefore, we can apply the item a) of Prop. 4.6, which (largely) suffices to tell us that the morphism $\widetilde{F}$ is triangulated.

It is clear that the functor $F \mapsto \widetilde{F}$ is a quasi-inverse to the functor $F \mapsto F \circ can$ since this is locally true for every diagram I. □

6. Derived extension of an exact category

In this section we work with derivators of type $\mathscr{D}ia_{\mathrm{f}}$, *i.e.*, derivators which are defined over the 2-subcategory of $\mathscr{D}ia$ that consists of finite directed diagrams.

Let us denote by $\mathsf{Mor}_{I^\circ}(\mathbb{T}(e))$ the category of I°-*morphisms* in $\mathbb{T}(e)$, *i.e.*, an object in this category is a family of morphisms of $\mathbb{T}(e)$

$$\{F_{\sigma(u),\tau(u)} : F_{\sigma(u)} \to F_{\tau(u)}\}_{u \in \mathsf{Mor}(I^\circ)} \quad \in \quad \prod_{j \in I} \prod_{i \in I} \prod_{I(j,i)} \mathsf{Hom}_{\mathbb{T}(e)}(F_i, F_j)$$

indexed over all arrows in I°, with the convention that $F_{\mathbf{1}_i} = \mathbf{1}_{F_i}$ for all $i \in I$. (It is possible that some morphism in the diagram is zero if in the category I there are objects with no

arrows in between.) A morphism in $\mathsf{Mor}_{I^\circ}(\mathbb{T}(e))$ is given by a family of morphisms

$$\{g_i : F_i \to G_i\}_{i\in I} \quad \in \quad \prod_{i\in I} \mathsf{Hom}_{\mathbb{T}(e)}(F_i, G_i)$$

such that the square

$$\begin{tikzcd} F_{\sigma(u)} \arrow[r, "F_{\sigma(u),\tau(u)}"] \arrow[d, "g_{\sigma(u)}"'] & F_{\tau(u)} \arrow[d, "g_{\tau(u)}"] \\ G_{\sigma(u)} \arrow[r, "G_{\sigma(u),\tau(u)}"'] & G_{\tau(u)} \end{tikzcd}$$

is commutative, for each arrow $u \in \mathsf{Mor}(I^\circ)$. It is clear that there is a natural isomorphism of categories

$$\underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e)) \xrightarrow{\simeq} \mathsf{Mor}_{I^\circ}(\mathbb{T}(e)) ,$$
$$F \mapsto \{F(u) : F(\sigma(u)) \to F(\tau(u))\}_{u\in\mathsf{Mor}(I^\circ)}$$

whose inverse is given by the functor which maps $\{F_{\sigma(u),\tau(u)}\}_{u\in\mathsf{Mor}(I^\circ)}$ to the presheaf F defined by $F(u) = F_{\sigma(u),\tau(u)}$, for all $u \in I^\circ$. Let us consider the functor

$$\mathbb{T}(I) \xrightarrow{\mathsf{mor}_I} \mathsf{Mor}_{I^\circ}(\mathbb{T}(e)) ,$$
$$X \mapsto \{\alpha^*_{\sigma(u),\tau(u)} X : \tau(u)^* X \to \sigma(u)^* X\}_{u\in\mathsf{Mor}(I)} ,$$

where, for each arrow $u \in \mathsf{Mor}(I)$, the natural transformation $\alpha^*_{\sigma(u),\tau(u)} : \tau(u)^* \to \sigma(u)^*$ is induced by the 2-arrow

$$\alpha_{\sigma(u),\tau(u)} : \sigma(u) \Rightarrow \tau(u) .$$

There is a commutative diagram

$$\begin{tikzcd} \underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e)) \arrow[r, "\simeq"] & \mathsf{Mor}_{I^\circ}(\mathbb{T}(e)) \\ \mathbb{T}(I). \arrow[u, "\mathsf{dia}_I"] \arrow[ur, "\mathsf{mor}_I"'] & \end{tikzcd}$$

and a canonical bijection

$$\mathsf{Hom}_{\underline{\mathsf{Hom}}(I^\circ,\mathbb{T}(e))}(\mathsf{dia}_I X, \mathsf{dia}_I Z) \tilde{\to} \mathsf{Hom}_{\mathsf{Mor}_{I^\circ}(\mathbb{T}(e))}(\mathsf{mor}_I X, \mathsf{mor}_I Z).$$

Let us prepare the crucial ingredients for the proof of the main theorem of this section.

6.1. **Full faithfulness of the diagram functor.** The following proposition gives a sufficient condition which allows us to lift morphisms of presheaves *uniquely*, when this is possible, from $\underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e))$ to morphisms of $\mathbb{T}(I)$. This will be helpful in the main theorem of this section in order to tell whether the functor dia_I is fully faithful when we consider its restriction to some subcategory of $\mathbb{T}(I)$ which enjoys this property (Toda condition).

Proposition 6.1. *Let $\mathbb{T}$ be a triangulated derivator of type $\mathscr{D}ia_{\mathsf{f}}$. Fix an arbitrary finite diagram I. Let X and Z be objects lying in $\mathbb{T}(I)$. Suppose that the (Toda) condition*

$$\mathsf{Hom}_{\mathbb{T}(e)}(\Sigma^n i^* X, j^* Z) = 0 , \qquad n > 0,$$

holds for each pair $(i,j) \in I \times I$. Then, the functor $\mathsf{dia}_I : \mathbb{T}(I) \to \underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e))$ induces a bijection

$$\mathsf{Hom}_{\mathbb{T}(I)}(X, Z) \xrightarrow{\sim} \mathsf{Hom}_{\underline{\mathsf{Hom}}(I^\circ,\mathbb{T}(e))}(\mathsf{dia}_I X, \mathsf{dia}_I Z).$$

Proof. Let us endow the additive category $\mathbb{T}(e)$ with an exact structure by defining conflations as the split short exact sequences. In this way, once a diagram I of $\mathscr{D}ia_{\mathsf{f}}$ has been fixed, we can consider $\underline{\mathsf{Hom}}(I^{\circ}, \mathbb{T}(e))$ as an exact category with the pointwise (split) exact structure.

Let X be an arbitrary object in the triangulated category $\mathbb{T}(I)$. For each arrow $a : j \to i$ in the diagram I, let $a_i \in \mathsf{Hom}_{\mathbb{T}(I)}(j_! i^* X, i_! i^* X)$ be the image of the identity $\mathbf{1}^{i_! i^* X}$ under the composition of homomorphisms

$$\begin{aligned}\mathsf{Hom}_{\mathbb{T}(I)}(i_! i^* X, i_! i^* X) &\tilde{\to} \mathsf{Hom}_{\mathbb{T}(e)}(i^* X, i^* i_! i^* X)\\ &\to \mathsf{Hom}_{\mathbb{T}(e)}(i^* X, j^* i_! i^* X)\\ &\tilde{\to} \mathsf{Hom}_{\mathbb{T}(I)}(j_! i^* X, i_! i^* X)\end{aligned}$$

induced by the adjoint pairs $i_! \dashv i^*$ and $j_! \dashv j^*$ and the 2-arrow α^*_{ji}

$$\begin{tikzcd} \mathbb{T}(I) \arrow[r, bend left, "j^*"] \arrow[r, bend right, "i^*"'] \arrow[r, phantom, "\alpha^*_{ji}\Uparrow"] & \mathbb{T}(e) \end{tikzcd}$$

associated with the natural transformation α_{ji}

$$\begin{tikzcd} e \arrow[r, bend left, "i"] \arrow[r, bend right, "j"'] \arrow[r, phantom, "\alpha_{ji}\Uparrow"] & I \end{tikzcd}$$

defined by the morphism a.

By using the formal properties of adjoint functors, we can find the formula for a_i

$$a_i = \varepsilon_j^{i_! i^* X} \circ j_! [\alpha^*_{ji}(i_! i^* X)] \circ j_! (\eta_i^{i^* X}) \,,$$

where, for each $l \in I$ and $Y \in \mathbb{T}(I)$, the morphisms ε_l^Y (resp., η_l^Y) are defined by using the counit (resp., the unit) of the adjunction $l_! \dashv l^*$.

Similarly, let $a_j \in \mathsf{Hom}_{\mathbb{T}(I)}(j_! i^* X, j_! j^* X)$ be the image of the identity $\mathbf{1}^{j_! j^* X}$ under the composition of homomorphisms

$$\begin{aligned}\mathsf{Hom}_{\mathbb{T}(I)}(j_! j^* X, j_! j^* X) &\tilde{\to} \mathsf{Hom}_{\mathbb{T}(e)}(j^* X, j^* j_! j^* X)\\ &\to \mathsf{Hom}_{\mathbb{T}(e)}(i^* X, j^* j_! j^* X)\\ &\tilde{\to} \mathsf{Hom}_{\mathbb{T}(I)}(j_! i^* X, j_! j^* X)\end{aligned}$$

induced by the adjoint pair $j_! \dashv j^*$ and the 2-arrow defined by the morphism a as above. Here, it is easy to see that the formula

$$a_j = j_! [\alpha^*_{ji}(X)]$$

for a_j holds. Thus, for each morphism $a : j \to i$ in the diagram I, we get an arrow

$$j_! i^* X \xrightarrow{[a_i, -a_j]^t} i_! i^* X \oplus j_! j^* X.$$

We claim that the composition of morphisms

$$j_! i^* X \xrightarrow{[a_i, -a_j]^t} i_! i^* X \oplus j_! j^* X \xrightarrow{[\varepsilon_i, \varepsilon_j]} X$$

is zero. Indeed, by 2-functoriality of α^*_{ji} there is the equality

$$j^*(\varepsilon_i) \circ \alpha^*_{ji}(i_! i^* X) = \alpha^*_{ji}(X) \circ i^*(\varepsilon_i).$$

Compose with the morphism $\eta_i^{i^*X}$ and use $i^*(\varepsilon_i)\circ\eta_i^{i^*X} = \mathbf{1}^{i^*X}$ to find the relation for α^*_{ji}

$$j^*(\varepsilon_i)\circ\alpha^*_{ji}(i_!i^*X)\circ\eta_i^{i^*X} = \alpha^*_{ji}(X).$$

Hence, we can compute

$$\begin{aligned}
[\varepsilon_i,\varepsilon_j]\circ[a_i,-a_j]^t &= \varepsilon_i\circ\varepsilon_j^{i_!i^*X}\circ j_![\alpha^*_{ji}(i_!i^*X)]\circ j_!(\eta_i^{i^*X}) - \varepsilon_j\circ j_![\alpha^*_{ji}(X)]\\
&= \varepsilon_j\circ j_![j^*(\varepsilon_i)]\circ j_![\alpha^*_{ji}(i_!i^*X)]\circ j_!(\eta_i^{i^*X}) - \varepsilon_j\circ j_![\alpha^*_{ji}(X)]\\
&= 0.
\end{aligned}$$

Here the second equality comes from the functoriality of ε_j and the third by the relation for α^*_{ji}.

Recall that the set of objects and the Hom-sets in I are finite and that the triangulated category $\mathbb{T}(I)$ has finite coproducts. So, we obtain a morphism in $\mathbb{T}(I)$

$$\textstyle\coprod_{I(j,i)} j_!i^*X \longrightarrow i_!i^*X\oplus j_!j^*X \xrightarrow{\ can\ } \coprod_{i\in I} i_!i^*X\ ,$$

whose components are $[a_i,-a_j]^t$. All these morphisms are the components of an arrow

$$\textstyle\coprod_{j\in I}\coprod_{i\in I-\{j\}}\coprod_{I(j,i)} j_!i^*X \xrightarrow{\ u\ } \coprod_{i\in I} i_!i^*X.$$

Now it is clear that the composition

$$\textstyle\coprod_{j\in I}\coprod_{i\in I-\{j\}}\coprod_{I(j,i)} j_!i^*X \xrightarrow{\ u\ } \coprod_{i\in I} i_!i^*X \xrightarrow{\ \varepsilon\ } X\ ,$$

where the morphism ε is induced by the ε_i, $i\in I$, still vanishes.

Let us call QX the object $\coprod_{i\in I} i_!i^*X$ and consider a distinguished triangle in $\mathbb{T}(I)$

$$LX \xrightarrow{\ v\ } QX \xrightarrow{\ \varepsilon\ } X \longrightarrow \Sigma LX.$$

Notice that the morphism $k^*(v)$ is the kernel of $k^*(\varepsilon)$, for any $k\in I$. Indeed, since the functor k^* is triangulated, we have a distinguished triangle in $\mathbb{T}(e)$

$$k^*(LX) \xrightarrow{\ k^*(v)\ } \textstyle\coprod_{i\in I}\coprod_{I(k,i)} i^*X \xrightarrow{\ k^*(\varepsilon)\ } k^*X \longrightarrow \Sigma k^*(LX).$$

The components of the morphism $k^*(\varepsilon)$ are

$$\alpha^*_{ki}(X) : i^*X \to k^*X\ ,$$

where α_{ki} is the 2-arrow

$$\begin{tikzcd} e \arrow[r, bend left, "i"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\alpha_{ki}\Uparrow"] & I \end{tikzcd}$$

defined by a morphism $k\to i$ in the diagram category I. Since in the diagram I there are no nontrivial loops, we have that $k^*(\varepsilon)$ is a section, with retraction given by a morphism

$$k^*X \longrightarrow \textstyle\coprod_{i\in I}\coprod_{I(k,i)} i^*X\ ,$$

all of whose components are zero but one, the identity on k^*X. It follows that $k^*(v)$ is actually the kernel of $k^*(\varepsilon)$ and that the connecting morphism ∂^{k^*X} must be zero. Hence, we have that locally the kernel is given by the equality

$$j^*(LX) = \coprod_{i\in I-\{j\}}\ \coprod_{I(j,i)} i^*X.$$

Now it is clear that

$$QLX = \coprod_{j\in I} j_!j^*LX = \coprod_{j\in I}\ \coprod_{i\in I-\{j\}}\ \coprod_{I(j,i)} j_!i^*X$$

is just the domain of the morphism u which we have constructed above.

Since we have seen that the composition $\varepsilon \circ u$ vanishes, there exists an arrow $\varepsilon^1 : QLX \to LX$ such that, by composition with v, it gives u, as in the following commutative diagram

$$\begin{tikzcd}
& QLX \arrow[dl, "\varepsilon^1"'] \arrow[d, "u"] \arrow[dr, "0"] & & \\
LX \arrow[r, "v"] & QX \arrow[r, "\varepsilon"] & X \arrow[r] & \Sigma LX.
\end{tikzcd}$$

By iterating this construction we get a 'resolution' as in the following commutative diagram in $\mathbb{T}(I)$

$$\begin{tikzcd}
QL^nX \arrow[rr, "u^n"] \arrow[dr, equal, "\varepsilon^n"'] & & QL^{n-1}X \arrow[r, "u^{n-1}"] \arrow[dr, "\varepsilon^{n-1}"'] & \dots \arrow[r, "u^2"] & QLX \arrow[rr, "u^1"] \arrow[dr, "\varepsilon^1"'] & & QX \arrow[r, "\varepsilon^0"] & X \\
& L^nX \arrow[ur, "v^n"'] & & \dots \arrow[ur, "v^2"'] & & LX \arrow[ur, "v^1"'] & &
\end{tikzcd}$$

Here, the maps ε^0, u^1, v^1 are ε, u and v, respectively. Moreover, compositions $u^k \circ u^{k+1}$ vanish, and all consecutive arrows v^{l+1}, ε^l fit in a distinguished triangle, for all l,

$$L^{l+1}X \xrightarrow{v^{l+1}} QL^lX \xrightarrow{\varepsilon^l} L^lX \longrightarrow \Sigma L^{l+1}X \ , \ l \in \mathbb{N} \, ,$$

such that $u^l = v^l \varepsilon^l$, for all $l \in \mathbb{N} - \{0\}$. In this sequence of maps we can easily check that

$$m^*(L^nX) \ = \coprod_{i_1 \neq m} \coprod_{I(m,i_1)} \cdots \coprod_{i_n \neq i_{n-1}} \coprod_{I(i_{n-1},i_n)} i_n^* X \, ,$$

for all $m \in I$. This shows that our 'resolution' of X must be finite since the diagram I is supposed to be finite.

If $l = n$ or $l = n-1$ we have something more, since in these cases distinguished triangles can be written as

$$QLW \xrightarrow{v} QW \xrightarrow{\varepsilon} W \longrightarrow \Sigma QLW \, ,$$

where W can be L^nX or $L^{n-1}X$. We recall that this means a distinguished triangle

$$\textstyle\coprod_{j\in I} \coprod_{i \in I-\{j\}} \coprod_{I(j,i)} j_! i^* W \xrightarrow{u} \coprod_{i\in I} i_! i^* W \xrightarrow{\varepsilon} W \longrightarrow \Sigma \coprod_{j\in I} \coprod_{i \in I-\{j\}} \coprod_{I(j,i)} j_! i^* W.$$

Let us apply the functor $\mathsf{Hom}_{\mathbb{T}(I)}(?, Z)$ to this distinguished triangle and get a long exact sequence of abelian groups

$$\begin{tikzcd}
\mathsf{Hom}_{\mathbb{T}(I)}(\Sigma \coprod_{j\in I} \coprod_{i \in I-\{j\}} \coprod_{I(j,i)} j_! i^* W, Z) \arrow[d] \\
\mathsf{Hom}_{\mathbb{T}(I)}(W, Z) \arrow[d, "\underline{\varepsilon}"] \\
\mathsf{Hom}_{\mathbb{T}(I)}(\coprod_{i\in I} i_! i^* W, Z) \arrow[d, "\underline{u}"] \\
\mathsf{Hom}_{\mathbb{T}(I)}(\coprod_{j\in I} \coprod_{i \in I-\{j\}} \coprod_{I(j,i)} j_! i^* W, Z).
\end{tikzcd}$$

Here, the morphisms $\underline{\varepsilon}$ and $\underline{u}$ are induced by composition from ε and u. By using the bijections induced by the adjunction pairs $j_! \dashv j^*$ and $i_! \dashv i^*$, we obtain an exact sequence

of abelian groups

$$\begin{tikzcd}
\prod_{j\in I}\prod_{i\in I-\{j\}}\prod_{I(j,i)} \mathsf{Hom}_{\mathbb{T}(e)}(\Sigma i^*W, j^*Z) \arrow[d] \\
\mathsf{Hom}_{\mathbb{T}(I)}(W,Z) \arrow[d, "\varepsilon'"] \\
\prod_{i\in I} \mathsf{Hom}_{\mathbb{T}(e)}(i^*W, i^*Z) \arrow[d, "u'"] \\
\prod_{j\in I}\prod_{i\in I-\{j\}}\prod_{I(j,i)} \mathsf{Hom}_{\mathbb{T}(e)}(i^*W, j^*Z).
\end{tikzcd}$$

Since this sequence is exact, it follows that the image of the homomorphism ε' precisely consists of the families of morphisms $\{f_i : i^*W \to i^*Z\}_{i\in I}$ which specify a morphism in the category $\mathsf{Mor}_{I^\circ}(\mathbb{T}(e))$. Indeed, if $u'(\{f_i\}_{i\in I})$ is zero, we have a commutative square

$$\begin{tikzcd}
i^*W \arrow[r, "{[\mathbf{1}^{i^*W}, -\alpha^*_{ij}W]^t}"] \arrow[d, "f_i"'] \arrow[dr, "0"] & i^*W \oplus j^*W \arrow[d, "f_i+f_j"] \\
i^*Z \arrow[r, "{[\mathbf{1}^{i^*Z}, -\alpha^*_{ij}Z]^t}"'] & i^*Z \oplus j^*Z \,,
\end{tikzcd}$$

for each arrow $a \in I(j,i)$. This clearly gives the equality $f_j \circ \alpha^*_{ij}(W) = \alpha^*_{ij}(Z) \circ f_i$, which is exactly the condition for the family $\{f_i\}_{i\in I}$ to be a morphism lying in the category $\mathsf{Mor}_{I^\circ}(\mathbb{T}(e))$.

Thus, we get an exact sequence

$$\begin{tikzcd}
\prod_{j\in I}\prod_{i\in I-\{j\}}\prod_{I(j,i)} \mathsf{Hom}_{\mathbb{T}(e)}(\Sigma i^*W, j^*Z) \arrow[d] \\
\mathsf{Hom}_{\mathbb{T}(I)}(W,Z) \arrow[d] \\
\mathsf{Hom}_{\mathsf{Mor}_{I^\circ}(\mathbb{T}(e))}(\mathsf{mor}_I W, \mathsf{mor}_I Z) \arrow[d] \\
0.
\end{tikzcd}$$

Notice that, passing in long exact sequence and using the Toda condition of the hypothesis, it is not difficult to see that there are vanishing groups

$$\mathsf{Hom}_{\mathbb{T}(e)}(\Sigma^n i^*W, j^*Z) = 0\,, \qquad n > 0,$$

for each pair $(i,j) \in I \times I$. This condition forces the first group in the sequence to be zero and we get a bijection

$$\mathsf{Hom}_{\mathbb{T}(I)}(W,Z) \xrightarrow{\simeq} \mathsf{Hom}_{\mathsf{Mor}_{I^\circ}(\mathbb{T}(e))}(\mathsf{mor}_I W, \mathsf{mor}_I Z).$$

Going from $l = n-2$ on, the situation is new. In this case a distinguished triangle is the horizontal one in the commutative diagram

$$\begin{tikzcd}
QL^{n-1}X \arrow[d, "\varepsilon^{n-1}"] \arrow[dr, "u^{n-1}"] & & & \Sigma QL^{n-1}X \arrow[d, "\Sigma\varepsilon^{n-1}"] \\
L^{n-1}X \arrow[r, "v^{n-1}"] & QL^{n-2}X \arrow[r, "\varepsilon^{n-2}"] & L^{n-2}X \arrow[r] & \Sigma L^{n-1}X.
\end{tikzcd}$$

After applying the functor $\mathsf{Hom}_{\mathbb{T}(I)}(?, Z)$ as in the initial step, we get a long exact sequence of abelian groups, that we report as the vertical sequence in the commutative diagram

$$\begin{tikzcd}
\mathsf{Hom}_{\mathbb{T}(I)}(\Sigma L^{n-1}X, Z) \arrow[r, hook, "\Sigma\underline{\varepsilon}^{n-1}"] \arrow[d] & \mathsf{Hom}_{\mathbb{T}(I)}(\Sigma QL^{n-1}X, Z) \\
\mathsf{Hom}_{\mathbb{T}(I)}(L^{n-2}X, Z) \arrow[d, "\underline{\varepsilon}^{n-2}"] & \\
\mathsf{Hom}_{\mathbb{T}(I)}(QL^{n-2}X, Z) \arrow[d, "\underline{v}^{n-1}"] \arrow[dr, "\underline{u}^{n-1}"] & \\
\mathsf{Hom}_{\mathbb{T}(I)}(L^{n-1}X, Z) \arrow[r, hook, "\underline{\varepsilon}^{n-1}"] & \mathsf{Hom}_{\mathbb{T}(I)}(QL^{n-1}X, Z).
\end{tikzcd}$$

By the preceding step we know that the maps $\underline{\varepsilon}^{n-1}$ and $\Sigma\underline{\varepsilon}^{n-1}$ are mono. This entails that the group $\mathsf{Hom}_{\mathbb{T}(I)}(\Sigma L^{n-1}X, Z)$ must be zero since the group $\mathsf{Hom}_{\mathbb{T}(I)}(\Sigma QL^{n-1}X, Z)$ is trivial. Indeed, using again the adjunction morphisms, we have

$$\begin{aligned}
\mathsf{Hom}_{\mathbb{T}(I)}(\Sigma QL^{n-1}X, Z) &= \mathsf{Hom}_{\mathbb{T}(I)}(\coprod_{i\in I} i_! i^* L^{n-1}X, Z) \\
&\tilde{\to} \prod_{i\in I} \mathsf{Hom}_{\mathbb{T}(I)}(i_! i^* L^{n-1}X, Z) \\
&\tilde{\to} \prod_{i\in I} \mathsf{Hom}_{\mathbb{T}(I)}(i^* L^{n-1}X, i^* Z) \\
&= 0.
\end{aligned}$$

The last equality comes from the condition found in the initial step. It follows that the homomorphism $\underline{\varepsilon}^{n-2}$ is a mono.

Moreover, the morphisms in $\mathsf{Hom}_{\mathbb{T}(I)}(QL^{n-2}X, Z)$ that are killed by the map $\underline{v}^{n-1}$ are exactly the same which are killed by $\underline{u}^{n-1}$. Hence, the homomorphism $\underline{\varepsilon}^{n-2} \circ \underline{v}^{n-1}$, which is zero, must factor, up to iso, through the group $\mathsf{Hom}_{\mathsf{Mor}_{I^\circ}(\mathbb{T}(e))}(\mathsf{mor}_I L^{n-2}X, \mathsf{mor}_I Z)$.

If we put everything together we find an exact sequence

$$0 \longrightarrow \mathsf{Hom}_{\mathbb{T}(I)}(L^{n-2}X, Z) \xrightarrow{\cong} \mathsf{Hom}_{\mathsf{Mor}_{I^\circ}(\mathbb{T}(e))}(\mathsf{mor}_I L^{n-2}X, \mathsf{mor}_I Z) \longrightarrow 0.$$

Now we can proceed by induction until we get an isomorphism

$$\mathsf{Hom}_{\mathbb{T}(I)}(X, Z) \xrightarrow{\cong} \mathsf{Hom}_{\mathsf{Mor}_{I^\circ}(\mathbb{T}(e))}(\mathsf{mor}_I X, \mathsf{mor}_I Z).$$

Clearly, this isomorphism is induced by the functor mor_I. Thus, we can use the canonical isomorphism between the categories $\underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e))$ and $\mathsf{Mor}_{I^\circ}(\mathbb{T}(e))$ and the claim follows. $\square$

6.2. **Epivalence of the diagram functor.** Suppose we are given a triangulated derivator $\mathbb{T}$ of type $\mathscr{D}ia$. Let us recall, from Appendix 1 in [11] to which we inspire, that, for each i in a small category I lying in $\mathscr{D}ia$, the evaluation functor of presheaves $F \mapsto F_i$ has a left adjoint denoted as $? \otimes i$,

$$\begin{tikzcd} \underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e)) \arrow[d, shift left, "(?)_i"] \\ \mathbb{T}(e) \arrow[u, shift left, "? \otimes i"] \end{tikzcd} .$$

For each $j \in I$, there is a canonical isomorphism

$$(F \otimes i)_j = \coprod_{I(j,i)} F.$$

Lemma 6.2 (Keller and Nicolàs, [11])**.** *Let I be any small category in $\mathscr{D}ia$. For each $i \in I$ the triangle*

$$\begin{tikzcd} \mathbb{T}(I) \arrow[r, "\mathsf{dia}_I"] & \underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e)) \\ \mathbb{T}(e) \arrow[u, "i_!"] \arrow[ur, "? \otimes i"'] & \end{tikzcd}$$

commutes up to a canonical isomorphism.

Remark 6.3. Everything we have said in this section is true if we replace the triangulated derivator $\mathbb{T}$ with the exact derivator $\underline{\mathbb{A}}$. In particular, Lemma 6.2 holds in this case. Moreover, suppose that $F_e : \underline{\mathbb{A}}(e) \to \mathbb{T}(e)$ is an exact functor and $\underline{F_{e_I}}$ is the induced functor defined on presheaves. It is not difficult to check that the diagram

$$\begin{tikzcd} \underline{\mathsf{Hom}}(I^\circ, \underline{\mathbb{A}}(e)) \arrow[r, "\underline{F_{e_I}}"] \arrow[d, shift left, "(?)_i"] & \underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e)) \arrow[d, shift left, "(?)_i"] \\ \underline{\mathbb{A}}(e) \arrow[r, "F_e"'] \arrow[u, shift left, "? \otimes i"] & \mathbb{T}(e) \arrow[u, shift left, "? \otimes i"] \end{tikzcd}$$

commutes in both directions.

The following proposition gives a sufficient condition which allows presheaves and their morphisms lift from $\underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e))$ to objects and morphisms of $\mathbb{T}(I)$. This tells us that the functor dia_I is an epivalence when we consider its restriction to the preimage of some subcategory of presheaves which enjoys this property (that we call 'Toda condition'). This result is one of the key ingredients in the proof of the main theorem of this section.

Proposition 6.4. *Let $\mathbb{T}$ be a triangulated derivator of type $\mathscr{D}ia_{\mathsf{f}}$. Fix an arbitrary finite diagram I. Then,*

a) *given an object F in $\underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e))$, such that the (Toda) condition*
$$\mathsf{Hom}_{\mathbb{T}(e)}(\Sigma^n F_i, F_j) = 0\,, \qquad n > 0,$$
holds for each pair $(i,j) \in I \times I$, there exists an object $\widetilde{F}$ in $\mathbb{T}(I)$ such that $\mathsf{dia}_I(\widetilde{F})$ is canonically isomorphic to F;

b) *given a morphism $f : F \to F'$ in $\underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e))$, such that the (Toda) conditions*
$$\mathsf{Hom}_{\mathbb{T}(e)}(\Sigma^n F_i, F_j) = 0\,,\ \mathsf{Hom}_{\mathbb{T}(e)}(\Sigma^n F'_i, F'_j) = 0\,,\ \mathsf{Hom}_{\mathbb{T}(e)}(\Sigma^n F_i, F'_j) = 0\,,$$
$n > 0$, hold for each pair $(i,j) \in I \times I$, there exists a morphism $\widetilde{f} : \widetilde{F} \to \widetilde{F'}$ in $\mathbb{T}(I)$ such that $\mathsf{dia}_I(\widetilde{f})$ is canonically isomorphic to f.

Proof. a) *Step 1. An exact category with finite global dimension.* Recall that every additive category can be endowed with an exact structure by considering as conflations the split exact pairs (*cf.* Example 2.9). In this way, we can consider $\mathbb{T}(e)$ as an exact category and endow $\underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e))$ with an exact structure by taking as conflations the pointwise split exact pairs.

Let us construct a projective-like resolution of the object F lying in $\underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e))$. As a first step we can consider the morphism

$$PF := \coprod_{i \in I} F_i \otimes i \xrightarrow{p^0} F$$

defined by using the counit of the adjunctions $? \otimes i \dashv (?)_i$. Clearly, the morphism p^0 is a deflation. Indeed, evaluating PX over j, we get a retraction in $\mathbb{T}(e)$

$$(PF)_j = \textstyle\coprod_{i \in I} \coprod_{I(j,i)} F_i \longrightarrow F_j.$$

We can form the (possibly non split) conflation in $\underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e))$

$$KF \overset{i^0}{\rightarrowtail} PF \overset{p^0}{\twoheadrightarrow} F.$$

It may happen that KF is already a projective-like object of type PKF and consequently the global dimension of the category $\underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e))$ is one even if the maximal length of the diagram I is infinite (*cf.* the case where I is $\mathbb{N}^\circ$ in [11, Appendix 1]). But in general this is not the case and we have to iterate the construction. Suppose that n is the maximal length of a chain of nonidentical arrows

$$i_n \longrightarrow \ldots \longrightarrow i_1 \longrightarrow i_0$$

in I. Then, it is not difficult to check by induction that the object $K^{n+1}F$ is zero. This fact shows that K^nF is projective and that the length of a projective resolution is bounded by n, as in the following chain complex

$$0 \longrightarrow PK^nF = K^nF \overset{i^{n-1}}{\rightarrowtail} PK^{n-1}F \xrightarrow{u^{n-1}} \ldots \longrightarrow PKF \xrightarrow{u^1} PF \overset{p^0}{\twoheadrightarrow} F.$$

Let us give a description of the morphisms in the above resolution. We can start with u^1. For each arrow $a^1 : j \to i$ in I, let $a^1_j \in \mathsf{Hom}(F_i \otimes j, F_j \otimes j)$ be the image of the identity $\mathbf{1}^{F_j \otimes j}$ under the composition of the morphisms

$$\begin{aligned}
\mathsf{Hom}_{\underline{\mathsf{Hom}}(I^\circ,\mathbb{T}(e))}(F_j \otimes j, F_j \otimes j) &\tilde{\to} \quad \mathsf{Hom}_{\mathbb{T}(e)}(F_j, (F_j \otimes j)_j) \\
&\to \quad \mathsf{Hom}_{\mathbb{T}(e)}(F_i, (F_j \otimes j)_j) \\
&\tilde{\to} \quad \mathsf{Hom}_{\underline{\mathsf{Hom}}(I^\circ,\mathbb{T}(e))}(F_i \otimes j, F_j \otimes j)
\end{aligned}$$

induced by the adjoint pair $? \otimes j \dashv (?)_j$ and the 2-arrow

$$\begin{tikzcd} \underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e)) \arrow[r, bend left, "(?)_i"] \arrow[r, bend right, "(?)_j"'] \arrow[r, phantom, "\Downarrow"] & \mathbb{T}(e) \end{tikzcd}$$

associated with the natural transformation

$$\begin{tikzcd} e \arrow[r, bend left, "j"] \arrow[r, bend right, "i"'] \arrow[r, phantom, "\Downarrow"] & I \end{tikzcd}$$

defined by the morphism a^1. Similarly, the morphism a^1 and the adjoint pair $? \otimes i \dashv (?)_i$ induce a morphism $a^1_i : F_i \otimes j \to F_i \otimes i$, image of $\mathbf{1}^{F_i \otimes i}$. Thus, for each morphism $a^1 : j \to i$ in I, we get an arrow

$$F_i \otimes j \xrightarrow{[a^1_i, -a^1_j]^t} F_i \otimes i \oplus F_j \otimes j \xrightarrow{can} \coprod_{i \in I} F_i \otimes i.$$

These are the components of the arrow

$$(KF)_j \otimes j = \coprod_{i \in I - \{j\}} \coprod_{I(j,i)} F_i \otimes j \xrightarrow{u^1_j} \coprod_{i \in I} F_i \otimes i$$

describing the morphism

$$PKF = \coprod_{j \in I} (KF)_j \otimes j \xrightarrow{u^1} PF.$$

We recall that here $I(j,i)$ denotes the discrete category of arrows $j \to i$ in I.

An explicit diagram, associated to a maximal length chain of nonidentical arrows in I,

$$i_n \longrightarrow \ldots \longrightarrow i_1 \longrightarrow i_0 \,,$$

which describes a part of the whole diagram associated to I, might help

$$\begin{tikzcd}
PKF \arrow[d] & 0 \arrow[r] \arrow[d] & (KF)_{i_1} \arrow[r] \arrow[d, equal] & (KF)_{i_2} \oplus (K^2F)_{i_2} \ldots \arrow[r] \arrow[d] & \ldots \\
KF \arrow[d, tail] & 0 \arrow[r] \arrow[d, tail] & (KF)_{i_1} \arrow[r] \arrow[d, tail] & (KF)_{i_2} \arrow[r] \arrow[d, tail] & \ldots \\
PF \arrow[d, two heads] & F_{i_0} \arrow[r] \arrow[d, equal] & F_{i_1} \oplus (KF)_{i_1} \arrow[r] \arrow[d] & F_{i_2} \oplus (KF)_{i_2} \arrow[r] \arrow[d] & \ldots \\
F & F_{i_0} \arrow[r] & F_{i_1} \arrow[r] & F_{i_2} \arrow[r] & \ldots
\end{tikzcd} .$$

By induction, one can get the description of all the maps in the projective resolution of F.

Step 2. Lifting a morphism of projective objects along the diagram functor. Define two objects in $\mathbb{T}(I)$

$$\widetilde{PKF} := \coprod_{j \in I} j_!(KF)_j = \coprod_{j \in I} \coprod_{i \in I - \{j\}} \coprod_{I(j,i)} j_!(F_i)$$

and

$$\widetilde{PF} := \coprod_{i \in I} i_!(F_i).$$

Lemma 6.2 tells us that these objects lift the diagrams PKF and PF along the functor dia_I. Let us construct the morphism $\widetilde{u}^1$ that lifts u^1. For each arrow $a^1 : j \to i$ in the diagram I, let $\widetilde{a^1}_i$ be the image of the identity $\mathbf{1}^{i_! F_i}$ under the composition of morphisms

$$\begin{aligned}
\mathsf{Hom}_{\mathbb{T}(I)}(i_! F_i, i_! F_i) \xrightarrow{\sim}\ & \mathsf{Hom}_{\mathbb{T}(e)}(F_i, i^* i_! F_i) \\
\to\ & \mathsf{Hom}_{\mathbb{T}(e)}(F_i, j^* i_! F_i) \\
\xrightarrow{\sim}\ & \mathsf{Hom}_{\mathbb{T}(I)}(j_! F_i, i_! F_i)
\end{aligned}$$

induced by the adjoint pairs $i_! \dashv i^*$ and $j_! \dashv j^*$ and the 2-arrow

$$\begin{tikzcd}
\mathbb{T}(I) \arrow[r, bend left, "i^*"] \arrow[r, bend right, "j^*"'] \arrow[r, phantom, "\Downarrow"] & \mathbb{T}(e)
\end{tikzcd}$$

associated with the natural transformation

$$\begin{tikzcd} e \arrow[r, bend left, "j"] \arrow[r, bend right, "i"'] \arrow[r, phantom, "\Downarrow"] & I \end{tikzcd}$$

defined by the morphism a^1. Let us denote by $\widetilde{a^1}_j$ the morphism $j_!(F_{a^1})$. Thus, for each morphism $a^1 : j \to i$ in I, we get an arrow

$$j_!F_i \xrightarrow{[\widetilde{a^1}_i, -\widetilde{a^1}_j]^t} i_!F_i \oplus j_!F_j \xrightarrow{\quad can \quad} \textstyle\coprod_{i\in I} i_!F_i .$$

These are the components of an arrow

$$j_!(KF)_j = \textstyle\coprod_{i\in I-\{j\}} \coprod_{I(j,i)} j_!F_i \xrightarrow{\quad \widetilde{u^1}_j \quad} \coprod_{i\in I} i_!F_i$$

describing the morphism

$$\widetilde{PKF} = \textstyle\coprod_{j\in I} j_!(KF)_j \xrightarrow{\quad \widetilde{u^1} \quad} \widetilde{PF}.$$

It is clear by the construction and by Lemma 6.2 that this morphism lifts u^1, *i.e.*, the morphism $\mathsf{dia}_I(\widetilde{u^1})$ is isomorphic to u^1.

Analogously, we can lift all the morphisms $u^l : PK^lF \to PK^{l-1}F$, $l \in \{1, \dots n\}$, of the projective resolution of F to morphisms $\widetilde{u^l} : \widetilde{PK^lF} \to \widetilde{PK^{l-1}}F$, $l \in \{1, \dots n\}$, in $\mathbb{T}(I)$.

Step 3. Lifting a projective resolution along the diagram functor. We construct a sequence of morphisms in $\mathbb{T}(I)$ which lifts the projective resolution that we have constructed in $\underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e))$,

$$0 \longrightarrow PK^nF = K^nF \xrightarrow{u^n} PK^{n-1}F \xrightarrow{u^{n-1}} \dots \xrightarrow{u^2} PKF \xrightarrow{u^1} PF \xrightarrow{u^0} F.$$

Recall that this is a strictly acyclic resolution, *i.e.*, there are conflations

$$K^{l+1}F \xrightarrow{i^l} PK^lF \xrightarrow{p^l} K^lF \ , \ l \in \mathbb{N} ,$$

such that $u^l = i^{l-1}p^l$, for all $l \in \mathbb{N} - \{0\}$.

We can start by lifting the morphism $u^n = i^{n-1}$ in the conflation

$$K^nF \xrightarrow{u^n} PK^{n-1}F \xrightarrow{p^{n-1}} K^{n-1}F$$

along the diagram functor to a morphism $\widetilde{u^n} : \widetilde{K^nF} \to \widetilde{PK^{n-1}}F$, following the step 2. Let us consider a distinguished triangle

$$\widetilde{K^nF} \xrightarrow{\widetilde{u^n}} \widetilde{PK^{n-1}}F \xrightarrow{\widetilde{p^{n-1}}} \widetilde{K^{n-1}}F \longrightarrow \Sigma\widetilde{K^nF} ,$$

which extends $\widetilde{u^n}$ in $\mathbb{T}(I)$. The image of the composition of morphisms $\widetilde{p^{n-1}} \circ \widetilde{u^n}$ under the functor dia_I is the zero morphism $\mathsf{dia}_I(\widetilde{p^{n-1}}) \circ u^n$. Therefore, there exists a *unique* morphism $\varphi^{n-1} : K^{n-1}F \to \mathsf{dia}_I(\widetilde{K^{n-1}F})$ such that $\mathsf{dia}_I(\widetilde{p^{n-1}})$ equals the composition $\varphi^{n-1} \circ p^{n-1}$.

For each $m \in I$, after applying the (triangulated) functor m^* to the distinguished triangle above, we get a distinguished triangle in $\mathbb{T}(e)$

$$(K^nF)_m \xrightarrow{\quad (u^n)_m \quad} (PK^{n-1}F)_m \xrightarrow{(\mathsf{dia}_I(\widetilde{p^{n-1}}))_m} (\mathsf{dia}_I(\widetilde{K^{n-1}}F))_m \longrightarrow \Sigma(K^nF)_m$$

which is split since u^n is an inflation. This shows that $(\mathsf{dia}_I(\widetilde{K^{n-1}F}))_m$ is the cokernel of $(u^n)_m$, for all $m \in I$. Thus, $(\varphi^{n-1})_m$ must be a (canonical) isomorphism, for all $m \in I$. It follows that $\mathsf{dia}_I(\widetilde{p^{n-1}})$ is the cokernel of u. Hence the canonical morphism φ^{n-1} is invertible.

Let us lift the morphism u^{n-1} along dia_I to a morphism $\widetilde{u^{n-1}}$ as in step 2. It is not difficult to verify, using the hypotheses by induction over $\mathbb{N}$, that

$$\mathsf{Hom}_{\mathbb{T}(e)}(\Sigma^n i^*(\widetilde{PK^{l+2}F}), j^*(\widetilde{PK^l F})) = 0\,, \qquad n > 0,$$

for all $l \in \mathbb{N}$. Since $\mathsf{dia}_I(\widetilde{u^{n-1}} \circ \widetilde{u^n})$ is zero, Proposition 6.1 with $l = n-2$ applies and tells us that the composition $\widetilde{u^{n-1}} \circ \widetilde{u^n}$ already vanishes in $\mathbb{T}(I)$. As a consequence, we get a morphism $\widetilde{i^{n-2}} : \widetilde{K^{n-1}F} \to \widetilde{PK^{n-2}F}$ such that the morphism $\widetilde{u^{n-1}} : \widetilde{PK^{n-1}F} \to \widetilde{PK^{n-2}F}$ equals the composition $\widetilde{i^{n-2}} \circ \widetilde{p^{n-1}}$, whose image under dia_I is the inflation $i^{n-2} : K^{n-1}F \to PK^{n-2}F$.

It is clear that we can iterate this construction until we get a distinguished triangle

$$\widetilde{KF} \xrightarrow{\widetilde{i^0}} \widetilde{PF} \xrightarrow{\widetilde{p^0}} \widetilde{F} \longrightarrow \Sigma\widetilde{KF}\,,$$

whose image under dia_I gives (up to a canonical iso) a conflation

$$KF \rightarrowtail^{i^0} PF \xrightarrow{p^0 = u^0} F.$$

Thus, by lifting a projective resolution of F, we have constructed an object $\widetilde{F}$ in the category $\mathbb{T}(I)$ which lifts F along the diagram functor dia_I.

b) *Step 1. Lifting a square of projective objects along the diagram functor.* Suppose that we are given a commutative square lying in $\underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e))$

$$\begin{tikzcd} PKF \arrow[r, "u"] \arrow[d, "h"] & PF \arrow[d, "g"] \\ PKF' \arrow[r, "u'"] & PF'. \end{tikzcd}$$

By looking at the structure of the objects PKF and PF (*cf.* with Step 1 in the proof of Claim a), we can see that the morphisms g and h are completely determined by their components, which are of the type $g_i \otimes j$ and $h_i \otimes j$, respectively. So, by Lemma 6.2, it is sufficient to lift these components to $j_!(g_i)$ and $j_!(h_i)$ and then reconstruct to obtain morphisms $\widetilde{h}$ and $\widetilde{g}$ which lift h and g, respectively.

Thanks to the hypotheses, we can use the second step in the proof of Claim a) of this proposition and lift the morphisms u and u' to the category $\mathbb{T}(I)$. We get a commutative square

$$\begin{tikzcd} \widetilde{PKF} \arrow[r, "\widetilde{u}"] \arrow[d, "\widetilde{h}"] & \widetilde{PF} \arrow[d, "\widetilde{g}"] \\ \widetilde{PKF'} \arrow[r, "\widetilde{u'}"] & \widetilde{PF'} \end{tikzcd}$$

in $\mathbb{T}(I)$ whose image under the diagram functor is canonically isomorphic to the given square.

To verify commutativity, let us use the hypotheses about F and F' in the same way as above. It is easy to check that the vanishing

$$\mathsf{Hom}_{\mathbb{T}(e)}(\Sigma^n i^*(\widetilde{PK^{l+2}F}), j^*(\widetilde{PK^l F'})) = 0\,, \qquad n > 0,$$

holds for all $l \in \mathbb{N}$. Since $\mathsf{dia}_I(\widetilde{g} \circ \widetilde{u} - \widetilde{u'} \circ \widetilde{h})$ is zero, Proposition 6.1 with $l = n - 2$ applies and tells us that the square commutes.

Step 2. Lifting a morphism of projective resolutions along the diagram functor. Let $f : F \to F'$ be an arbitrary morphism in the category $\underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e))$. We know from the proof of Claim a) that we can construct projective resolutions of F and F' whose lengths are bounded by the maximal length of a chain of nonidentical arrows in I,

$$0 \longrightarrow PK^nF = K^nF \overset{u^n}{\rightarrowtail} PK^{n-1}F \xrightarrow{u^{n-1}} \dots \xrightarrow{u^2} PKF \xrightarrow{u^1} PF \overset{u^0}{\twoheadrightarrow} F$$

and

$$0 \longrightarrow PK^nF' = K^nF' \overset{u'^n}{\rightarrowtail} PK^{n-1}F' \xrightarrow{u'^{n-1}} \dots \xrightarrow{u'^2} PKF' \xrightarrow{u'^1} PF' \overset{u'^0}{\twoheadrightarrow} F'.$$

Since these resolutions are made of projective objects, the morphism $f : F \to F'$ extends to a morphism of resolutions lying in $\underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e))$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & PK^nF = K^nF \arrow[r, tail, "u^n"] \arrow[d, "f^n"] & PK^{n-1}F \arrow[r, "u^{n-1}"] \arrow[d, "f^{n-1}"] & \dots \arrow[r, "u^2"] & PKF \arrow[r, "u^1"] \arrow[d, "f^1"] & PF \arrow[r, two heads, "u^0"] \arrow[d, "f^0"] & F \arrow[d, "f"] \\
0 \arrow[r] & PK^nF' = K^nF' \arrow[r, tail, "u'^n"] & PK^{n-1}F' \arrow[r, "u'^{n-1}"] & \dots \arrow[r, "u'^2"] & PKF' \arrow[r, "u'^1"] & PF' \arrow[r, two heads, "u'^0"] & F'.
\end{tikzcd}$$

Let us start by lifting the square of projective objects on the left side of the commutative diagram

$$\begin{tikzcd}
K^nF \arrow[r, tail, "u^n"] \arrow[d, "f^n"] & PK^{n-1}F \arrow[r, two heads, "p^{n-1}"] \arrow[d, "f^{n-1}"] & K^{n-1}F \arrow[d, "g^{n-1}"] \\
K^nF' \arrow[r, tail, "u'^n"] & PK^{n-1}F' \arrow[r, two heads, "p'^{n-1}"] & K^{n-1}F' ,
\end{tikzcd}$$

where the morphism g^{n-1} is induced by the universal property of the cokernel. It is routine to verify that the sufficient conditions which allow us using the first step of this claim hold. We get a commutative square

$$\begin{tikzcd}
\widetilde{K^nF} \arrow[r, "\widetilde{u^n}"] \arrow[d, "\widetilde{f^n}"] & \widetilde{PK^{n-1}F} \arrow[d, "\widetilde{f^{n-1}}"] \\
\widetilde{K^nF'} \arrow[r, "\widetilde{u'^n}"] & \widetilde{PK^{n-1}F'} ,
\end{tikzcd}$$

whose image under the diagram functor is canonically isomorphic to the given one.

Let us consider an extension in $\mathbb{T}(I)$ of the latter square to a morphism of distinguished triangles

$$\begin{tikzcd}
\widetilde{K^nF} \arrow[r, "\widetilde{u^n}"] \arrow[d, "\widetilde{f^n}"] & \widetilde{PK^{n-1}F} \arrow[r, "\widetilde{p^{n-1}}"] \arrow[d, "\widetilde{f^{n-1}}"] & \widetilde{K^{n-1}F} \arrow[r] \arrow[d, "\widetilde{g^{n-1}}"] & \Sigma\widetilde{K^nF} \arrow[d, "\Sigma\widetilde{f^n}"] \\
\widetilde{K^nF'} \arrow[r, "\widetilde{u'^n}"] & \widetilde{PK^{n-1}F'} \arrow[r, "\widetilde{p'^{n-1}}"] & \widetilde{K^{n-1}F'} \arrow[r] & \Sigma\widetilde{K^nF'}.
\end{tikzcd}$$

The images of the compositions of morphisms $\widetilde{p^{n-1}} \circ \widetilde{u^n}$ and $\widetilde{p'^{n-1}} \circ \widetilde{u'^n}$ under the functor dia_I are the zero morphisms $\mathsf{dia}_I(\widetilde{p^{n-1}}) \circ u^n$ and $\mathsf{dia}_I(\widetilde{p'^{n-1}}) \circ u'^n$, respectively. It follows that there exist *unique* morphisms $\varphi^{n-1} : K^{n-1}F \to \mathsf{dia}_I(\widetilde{K^{n-1}F})$ and $\varphi'^{n-1} : K^{n-1}F' \to \mathsf{dia}_I(\widetilde{K^{n-1}F'})$ such that $\mathsf{dia}_I(\widetilde{p^{n-1}})$ and $\mathsf{dia}_I(\widetilde{p'^{n-1}})$ respectively equal compositions $\varphi^{n-1} \circ p^{n-1}$ and $\varphi'^{n-1} \circ p'^{n-1}$. Moreover, there is an isomorphism $\mathsf{dia}_I(\widetilde{g^{n-1}}) \circ \varphi^{n-1} = \varphi'^{n-1} \circ g^{n-1}$.

For each $m \in I$, we apply the (triangulated) functor m^* to the morphism of distinguished triangles above and get a morphism of distinguished triangles in $\mathbb{T}(e)$

$$\begin{tikzcd}
(K^nF)_m \arrow[r, "(u^n)_m"] \arrow[d, "(f^n)_m"] & (PK^{n-1}F)_m \arrow[r, "(\mathsf{dia}_I(\widetilde{p^{n-1}}))_m"] \arrow[d, "(f^{n-1})_m"] & (\mathsf{dia}_I(\widetilde{K^{n-1}F}))_m \arrow[r] \arrow[d, "(\mathsf{dia}_I(\widetilde{g^{n-1}}))_m"] & \Sigma(K^nF)_m \arrow[d, "\Sigma(f^n)_m"] \\
(K^nF')_m \arrow[r, "(u'^n)_m"] & (PK^{n-1}F')_m \arrow[r, "(\mathsf{dia}_I(\widetilde{p'^{n-1}}))_m"] & (\mathsf{dia}_I(\widetilde{K^{n-1}F'}))_m \arrow[r] & \Sigma(K^nF')_m .
\end{tikzcd}$$

Here, distinguished triangles are split since u^n and u'^n are inflations. This shows that the morphisms $(\mathsf{dia}_I(\widetilde{K^{n-1}F}))_m$ and $(\mathsf{dia}_I(\widetilde{K^{n-1}F'}))_m$ are the cokernels of $(u^n)_m$ and $(u'^n)_m$, for all $i \in I$. Thus, $(\varphi^{n-1})_m$ and $(\varphi'^{n-1})_m$ must be (canonical) isomorphisms, for all $m \in I$. It follows that the morphisms $\mathsf{dia}_I(\widetilde{p^{n-1}})$ and $\mathsf{dia}_I(\widetilde{p'^{n-1}})$ are the cokernels of u^n and u'^n. Hence, the morphisms φ^{n-1} and φ'^{n-1} are invertible. Finally, the universal property of cokernels induces a (canonical) isomorphism from g^{n-1} into $\mathsf{dia}_I(\widetilde{g^{n-1}})$.

At this point we can proceed by lifting the square of projective objects

$$\begin{tikzcd}
PK^{n-1}F \arrow[r, "u^{n-1}"] \arrow[d, "f^{n-1}"] & PK^{n-2}F \arrow[d, "f^{n-2}"] \\
PK^{n-1}F' \arrow[r, "u'^{n-1}"] & PK^{n-2}F' ,
\end{tikzcd}$$

according to the Step 1 of this Claim b), in order to get a commutative square

$$\begin{tikzcd}
\widetilde{PK^{n-1}F} \arrow[r, "\widetilde{u^{n-1}}"] \arrow[d, "\widetilde{f^{n-1}}"] & \widetilde{PK^{n-2}F} \arrow[d, "\widetilde{f^{n-2}}"] \\
\widetilde{PK^{n-1}F'} \arrow[r, "\widetilde{u'^{n-1}}"] & \widetilde{PK^{n-2}F'} ,
\end{tikzcd}$$

whose image under the diagram functor is isomorphic to the given one.

Sufficient conditions in order to apply Proposition 6.1 hold. Since the images $\mathsf{dia}_I(\widetilde{u^{n-1}} \circ \widetilde{u^n})$ and $\mathsf{dia}_I(\widetilde{u'^{n-1}} \circ \widetilde{u'^n})$ are both isomorphic to zero, it follows that the compositions $\widetilde{u^{n-1}} \circ \widetilde{u^n}$ and $\widetilde{u'^{n-1}} \circ \widetilde{u'^n}$ already vanish in $\mathbb{T}(I)$. Therefore, there exist morphisms $\widetilde{i^{n-2}}$ and $\widetilde{i'^{n-2}}$, whose images are the inflations i^{n-2} and i'^{n-2}, such that $\widetilde{u^{n-1}}$ and $\widetilde{u'^{n-1}}$ are respectively isomorphic to the compositions $\widetilde{i^{n-2}} \circ \widetilde{p^{n-1}}$ and $\widetilde{i'^{n-2}} \circ \widetilde{p'^{n-1}}$.

It is clear that we can continue by iterating this construction until the cohomological degree is 0. In the end, we will get a morphism $\widetilde{f} : \widetilde{F} \to \widetilde{F'}$, which lifts the given morphism $f : F \to F'$ to the triangulated category $\mathbb{T}(I)$ along the diagram functor dia_I. □

6.3. **Invertibility of the diagram functor.** Let us recall that a (triangulated) category (resp., a (triangulated) functor) is *basic* if it occurs as the evaluation over the one object diagram e of a (triangulated) derivator (resp., of a morphism of (triangulated) derivators). Apparently, it is not known whether all (triangulated) categories are basic, nor if the 'extension' of a (triangulated) category to a (triangulated) derivator having it as a base is unique. However, we have some reasons to think of a negative answer to both of these statements.

We want to stress that 'basic' always implies a *choice* of a derivator. As a consequence, when we claim that some basic morphism extends 'uniquely', we always mean 'uniquely with respect to a *chosen* derivator'.

Let us also recall that we say that an additive functor $F : \mathcal{E} \to \mathcal{T}$ from an exact category $\mathcal{E}$ into a triangulated category $\mathcal{T}$ is *exact* or, equivalently, a ∂-*functor* if to any conflation

in $\mathcal{E}$,

$$X \rightarrowtail^{u} Y \xrightarrow{v} \!\!\!\!\rightarrow Z\,,$$

it *functorially* associates a morphism $\partial(\varepsilon)$ such that the diagram

$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{\partial(\varepsilon)} \Sigma F(X)$$

is a distinguished triangle of $\mathcal{T}$. It is straightforward to see that an additive morphism $\mu : F \to F'$ of exact functors is an *exact* morphism if to the conflation above it *functorially* associates a morphism of distinguished triangles of $\mathcal{T}$,

$$\begin{tikzcd}
F(X) \arrow[r, "F(u)"] \arrow[d, "\mu^X"'] & F(Y) \arrow[r, "F(v)"] \arrow[d, "\mu^Y"'] & F(Z) \arrow[r, "\partial(\varepsilon)"] \arrow[d, "\mu^Z"] & \Sigma F(X) \arrow[d, "\Sigma\mu^X"] \\
F'(X) \arrow[r, "F'(u)"] & F'(Y) \arrow[r, "F'(v)"] & F'(Z) \arrow[r, "\partial'(\varepsilon)"] & \Sigma F'(X)\,.
\end{tikzcd}$$

In any case, let us remark that, in the presence of the Toda condition as in the items a) and b) of the following theorem, it would be equivalent if the basic exact morphism μ of basic exact functors F, F' that we want to extend were taken *weakly* exact only. Indeed, it is not difficult to see that *all* the weakly exact functors are exact in the presence of such a condition.

Theorem 6.5. *Let $\mathcal{A}$ be an exact category and $\mathcal{T}$ a triangulated category such that there exists an isomorphism $\mathcal{T} \tilde{\to} \mathbb{T}(e)$ for some triangulated derivator $\mathbb{T}$ of type $\mathscr{D}ia_{\mathsf{f}}$.*

a) *Suppose that $F : \mathcal{A} \to \mathcal{T}$ is an exact functor such that the (Toda) condition*

$$\mathsf{Hom}_{\mathcal{T}}(\Sigma^n F(X), F(Y)) = 0\,, \qquad n > 0,$$

holds for all X, Y in $\mathcal{A}$. Then, there exists an exact morphism of derivators $\widetilde{F} : \underline{\mathbb{A}} \to \mathbb{T}$ (of type $\mathscr{D}ia_{\mathsf{f}}$) having base F (up to iso). The morphism $\widetilde{F}$ is unique up to a unique isomorphism.

b) *Suppose that F and F' are exact morphisms $\underline{\mathbb{A}} \to \mathbb{T}$ and that the (Toda) conditions*

$$\mathsf{Hom}_{\mathbb{T}(e)}(\Sigma^n F_e(X), F_e(Y)) = 0\,, \qquad n > 0\,,$$

$$\mathsf{Hom}_{\mathbb{T}(e)}(\Sigma^n F'_e(X), F'_e(Y)) = 0\,, \qquad n > 0\,,$$

$$\mathsf{Hom}_{\mathbb{T}(e)}(\Sigma^n F_e(X), F'_e(Y)) = 0\,, \qquad n > 0\,,$$

hold for all X, Y in $\underline{\mathbb{A}}(e) = \mathcal{A}$. Then, the map

$$\mathsf{Hom}_{\underline{\mathcal{H}om}_{ex}(\underline{\mathbb{A}},\mathbb{T})}(F, F') \longrightarrow \mathsf{Hom}_{\underline{\mathsf{Hom}}_{ex}(\mathcal{A},\mathbb{T}(e))}(F_e, F'_e)\,, \qquad \mu \mapsto \mu_e$$

is bijective.

Proof. a) For any diagram I in $\mathscr{D}ia_{\mathsf{f}}$, let us denote by

$$\underline{F}_I : \underline{\mathsf{Hom}}(I^\circ, \underline{\mathbb{A}}(e)) \longrightarrow \underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e))$$

the induced functor defined on presheaves. Let $\mathbb{V}(I)$ be the essential image of the functor $\underline{F}_I$, which is an additive subcategory of $\underline{\mathsf{Hom}}(I^\circ, \underline{\mathbb{A}}(e))$. Because of the hypotheses of claim a), every object lying in $\mathbb{V}(I)$ clearly satisfies the hypotheses of claim a) of Proposition 6.4. Therefore, according to that claim they lift to $\mathbb{T}(I)$. Thus, we can form an additive category $\mathbb{U}(I)$ by taking all the preimages under dia_I of all the objects lying in $\mathbb{V}(I)$ and then considering the additive full subcategory they span in $\mathbb{T}(I)$.

By construction, the restriction of the diagram functor dia_I to $\mathbb{U}(I)$ is essentially surjective. Actually, it is fully faithful, too. Indeed, for every pair of objects X and Z in $\mathbb{U}(I)$, the hypotheses of Proposition 6.1 hold. Hence, the conclusion gives us a bijection

$$\mathsf{Hom}_{\mathbb{T}(I)}(X,Z) \xrightarrow{\sim} \mathsf{Hom}_{\underline{\mathsf{Hom}}(I^\circ,\mathbb{T}(e))}(\mathsf{dia}_I X, \mathsf{dia}_I Z)\,,$$

induced by dia_I, which clearly reduces to a bijection

$$\mathsf{Hom}_{\mathbb{U}(I)}(X,Z) \xrightarrow{\sim} \mathsf{Hom}_{\mathbb{V}(I)}(\mathsf{dia}_I X, \mathsf{dia}_I Z)\,,$$

induced by its restriction to the subcategory $\mathbb{U}(I)$, for all pairs of objects X and Z in $\mathbb{U}(I)$.

Said otherwise, the restriction of the diagram functor to $\mathbb{U}(I)$ is an (additive) equivalence of (additive) categories. Nevertheless, we will show that its (inverse) composition with $\underline{F}_I \circ \mathsf{dia}_I^{\mathbb{A}}$ gives rise to a ∂-functor from $\mathbb{A}(I)$ to $\mathbb{T}(I)$.

Consider the following commutative diagram of additive categories

$$\begin{tikzcd}
\underline{\mathsf{Hom}}(I^\circ, \underline{\mathbb{A}}(e)) \arrow[r] & \mathbb{V}(I) \arrow[r] & \underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e)) \\
\underline{\mathbb{A}}(I) \arrow[u, "\cong"] \arrow[r, dashed] & \mathbb{U}(I) \arrow[u, "\simeq"'] \arrow[r] & \mathbb{T}(I) \arrow[u, "\mathsf{dia}_I"']
\end{tikzcd}$$

that we have constructed for any diagram I in $\mathscr{D}ia_{\mathsf{f}}$. The dashed arrow is obtained by composition with the functor which is inverse to the vertical equivalence in the centre of the diagram. It clearly induces an additive functor $\widetilde{F}_I : \underline{\mathbb{A}}(I)| \to \mathbb{T}(I)|$ for every diagram I lying in $\mathscr{D}ia_{\mathsf{f}}$. Here and in the sequel, the symbol $|$ means the image under the forgetful functor.

This construction gives rise to a diagram of *additive derivators*, *i.e.*, derivators whose image is in the subcategory $\mathscr{A}dd$ of additive categories,

$$\underline{\mathbb{A}}| \longrightarrow \mathbb{V} \xleftarrow{\sim} \mathbb{U} \longrightarrow \mathbb{T}|.$$

After inverting the equivalence and composing, we get an additive morphism of additive derivators $\widetilde{F} : \underline{\mathbb{A}}| \to \mathbb{T}|$. It is indeed easy to check that, for all the morphisms $u : I \to J$ in $\mathscr{D}ia_{\mathsf{f}}$, we have an isomorphism of functors $u^*\widetilde{F}_J \xrightarrow{\sim} \widetilde{F}_I u^*$. Moreover, it is clear that the base of $\widetilde{F}$ is naturally isomorphic to the functor F as an *additive* functor.

We want to show that, with respect to the exact and triangulated structures of the categories $\mathbb{A}(I)$ and $\mathbb{T}(I)$, the functor $\widetilde{F}_I$ is weakly exact according to definition 4.3. Suppose we are given a pair of morphisms in $\mathbb{U}(I)$

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

which is the image under $\widetilde{F}_I$ of some conflation ε lying in $\underline{\mathbb{A}}(I)$. We can extend the morphism f to a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{h} W \xrightarrow{l} \Sigma X$$

in the triangulated category $\mathbb{T}(I)$. Since the composition of f with g vanishes there is an arrow $\varphi : W \to Z$ which lifts g. After applying to this distinguished triangle the (triangulated) functor i^*, we get a distinguished triangle in $\mathbb{T}(e)$

$$i^*X \xrightarrow{i^*f} i^*Y \xrightarrow{i^*h} i^*W \xrightarrow{\delta_i l} \Sigma i^*X\,,$$

for any $i \in I$. As the morphisms f and g are in the image of a conflation of $\mathbb{A}(I)$ and $F : \mathcal{A} \to \mathcal{T}$ is supposed to be a ∂-functor, we get that this triangle fits in a morphism of

distinguished triangles

$$\begin{tikzcd} i^*X \arrow[r, "i^*f"] \arrow[d, equal] & i^*Y \arrow[r, "i^*h"] \arrow[d, equal] & i^*W \arrow[r, "\delta_i l"] \arrow[d, "\psi_i", "\wr"'] & \Sigma i^*X \arrow[d, equal] \\ i^*X \arrow[r, "i^*f"] & i^*Y \arrow[r, "i^*g"] & i^*Z \arrow[r, "m_i"] & \Sigma i^*X. \end{tikzcd}$$

Here, the invertible arrow $\psi_i : i^*W \to i^*Z$ which makes the diagram commute exists thanks to the axioms of triangulated categories. Since the objects X and Z belong to the subcategory $\mathbb{U}(I)$, their images under the (restriction of the) functor dia_I are in $\mathbb{V}(I)$.

Hence, we know from the Toda condition that the abelian group $\mathsf{Hom}_{\mathbb{T}(e)}(\Sigma i^*X, i^*Z)$ must be zero for each $i \in I$. This condition makes clear that the isomorphism ψ_i is uniquely determined. It follows that ψ_i must be canonically isomorphic to $i^*\varphi$, for all $i \in I$. Now we can use Axiom **Der 2** of derivators which ensures that φ actually is an isomorphism. This shows that the pair of morphisms $(f\ ,\ g)$ actually extends to a distinguished triangle of $\mathbb{T}(I)$

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{m} \Sigma X\ ,$$

for *some* morphism m.

Now, since we have checked that the hypotheses of Proposition 4.6 b) are fulfilled, we know that the morphism $\widetilde{F} : \underline{\mathbb{A}} \to \mathbb{T}$ has the *property* of being an exact morphism of derivators. We have already checked at the beginning of the proof of 4.6 b) that the base of our morphism $\widetilde{F}$ is canonically an exact functor. Therefore, it must coincide (up to a canonical iso) with F as an *exact* functor. The uniqueness of our construction also follows by the Toda condition.

b) We want to show that the functor

$$\underline{\mathcal{H}om}_{ex}(\underline{\mathbb{A}}, \mathbb{T}) \xrightarrow{\quad \mathsf{ev}_e \quad} \underline{\mathsf{Hom}}_{ex}(\underline{\mathbb{A}}(e), \mathbb{T}(e)), \qquad F \mapsto F_e$$

induces a bijection

$$\mathsf{Hom}_{\underline{\mathcal{H}om}_{ex}(\underline{\mathbb{A}},\mathbb{T})}(F, F') \xrightarrow{\ \sim\ } \mathsf{Hom}_{\underline{\mathsf{Hom}}_{ex}(\underline{\mathbb{A}}(e),\mathbb{T}(e))}(F_e, F'_e).$$

It is known (cf. [8, 8.1]) the easy fact that, under the hypotheses of b) (Toda conditions), the set of morphisms of ∂-functors from F_e to F'_e is in bijection with the set of morphisms of their underlying additive functors. Moreover, after item b) of Prop. 4.6 we know that the set of morphisms of exact morphisms from F to F' is in bijection with the subset of morphisms of their underlying additive morphisms from $F|$ to $F'|$.

Thus, to prove the claim in item b) of this theorem, we only have to show that the functor

$$\underline{\mathcal{H}om}_{add}(\underline{\mathbb{A}}|, \mathbb{T}|) \xrightarrow{\quad \mathsf{ev}|_e \quad} \underline{\mathsf{Hom}}_{add}(\underline{\mathbb{A}}(e)|, \mathbb{T}(e)|), \qquad F| \mapsto F|_e$$

induces a bijection

$$\mathsf{Hom}_{\underline{\mathcal{H}om}_{add}(\underline{\mathbb{A}}|,\mathbb{T}|)}(F|, F'|) \xrightarrow{\ \sim\ } \mathsf{Hom}_{\underline{\mathsf{Hom}}_{add}(\underline{\mathbb{A}}(e)|,\mathbb{T}(e)|)}(F|_e, F'|_e).$$

From now on we omit the symbol $|$. Let us factor the functor ev_e as follows

$$\begin{tikzcd} \underline{\mathcal{H}om}_{add}(\underline{\mathbb{A}}, \mathbb{T}) \arrow[rr, "\mathsf{ev}_e"] \arrow[dr, "\underline{\mathsf{dia}}_{\mathbb{A}}"'] & & \underline{\mathsf{Hom}}_{add}(\underline{\mathbb{A}}(e), \mathbb{T}(e)) \\ & \underline{\mathcal{H}om}_{add}(\underline{\mathbb{A}}, \underline{\mathbb{T}(e)}) \arrow[ur, "\sim"'] & \end{tikzcd} .$$

Here, $\underline{\mathbb{T}(e)}$ is the additive derivator which associates the additive category $\underline{\mathsf{Hom}}(I^\circ, \mathbb{T}(e))$ with a diagram $I \in \mathscr{D}ia_{\mathsf{f}}$. The additive category $\underline{\mathcal{H}om_{add}}(\underline{\mathbb{A}}, \underline{\mathbb{T}(e)})$ contains as objects the morphisms of the additive underlying derivators, *i.e.*, morphisms $F : \underline{\mathbb{A}} \to \underline{\mathbb{T}(e)}$ such that F_I are additive functors for all diagrams $I \in \mathscr{D}ia_{\mathsf{f}}$ with compatibility conditions. The additive functor $\underline{\mathsf{dia}_{\mathbb{A}}}$ is induced by the morphism of the additive underlying derivators $\mathsf{dia} : \mathbb{T} \to \underline{\mathbb{T}(e)}$ by composition. Moreover, the additive functor of additive categories on the right of the last diagram above is given by evaluation on the one object diagram e. It is not hard to directly check that it is an equivalence of categories.

Thus, in order to prove the claim, it is enough to show that the morphism dia induces a bijection

$$\mathsf{Hom}_{\underline{\mathcal{H}om}_{add}(\underline{\mathbb{A}},\mathbb{T})}(F, F') \xrightarrow{\sim} \mathsf{Hom}_{\underline{\mathcal{H}om}_{add}(\underline{\mathbb{A}},\underline{\mathbb{T}(e)})}(\mathsf{dia} \circ F, \mathsf{dia} \circ F').$$

By the contravariant version of Lemma A.5 in [9], we can check this isomorphism locally, *i.e.*, we have to show that the functor $\mathsf{dia}_I : \mathbb{T}(I) \to \underline{\mathbb{T}(e)}(I)$ induces a bijection

$$\begin{array}{c} \mathsf{Hom}_{\underline{\mathsf{Hom}}_{add}(\underline{\mathbb{A}}(J),\mathbb{T}(I))}(F_I \circ u^*, F'_I \circ u^*) \\ \Big\downarrow \wr \\ \mathsf{Hom}_{\underline{\mathsf{Hom}}_{add}(\underline{\mathbb{A}}(J),\underline{\mathbb{T}(e)}(I))}((\mathsf{dia} \circ F)_I \circ u^*, (\mathsf{dia} \circ F')_I \circ u^*)\,, \end{array}$$

for each 1-morphism $u : I \to J$ in $\mathscr{D}ia_{\mathsf{f}}$. This is *a posteriori* true if the map

$$\mathsf{Hom}_{\mathbb{T}(I)}(\Sigma^n F_I X, F'_I Y) \longrightarrow \mathsf{Hom}_{\underline{\mathbb{T}(e)}(I)}(\Sigma^n \mathsf{dia}_I(F_I X), \mathsf{dia}_I(F'_I Y))$$

is a bijection for all X, Y in $\underline{\mathbb{A}}(I)$. But this is true, since, by the argument of the proof of Proposition 6.1, we know that, for all $n \in \mathbb{N}$, there is an exact sequence

$$\begin{array}{c} \prod_{j\in I} \prod_{i \in I-\{j\}} \prod_{I^\circ(i,j)} \mathsf{Hom}_{\mathbb{T}(e)}(\Sigma^{n+1}(\mathsf{dia}_I(F_I X))_i, (\mathsf{dia}_I(F'_I Y))_j) \\ \Big\downarrow \\ \mathsf{Hom}_{\mathbb{T}(I)}(\Sigma^n F_I X, F'_I Y) \\ \Big\downarrow \\ \mathsf{Hom}_{\underline{\mathbb{T}(e)}(I)}(\Sigma^n \mathsf{dia}_I(F_I X), \mathsf{dia}_I(F'_I Y)) \\ \Big\downarrow \\ 0\,, \end{array}$$

where the first group vanishes by the third Toda condition in the hypotheses. The assertion follows. $\square$

This theorem, combined with Theorem 2.13 immediately yields Theorem 2.14 as a corollary.

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